10 Chapter Summary

What did you learn?

Section 10.1	Review Exercises
☐ Find the inclination of a line *(p. 728)*.	1–4
☐ Find the angle between two lines *(p. 729)*.	5–8
☐ Find the distance between a point and a line *(p. 730)*.	9, 10
Section 10.2	
☐ Recognize a conic as the intersection of a plane and a double-napped cone *(p. 735)*.	11, 12
☐ Write equations of parabolas in standard form and graph parabolas *(p. 736)*.	13–16
☐ Use the reflective property of parabolas to solve real-life problems *(p. 738)*.	17–20
Section 10.3	
☐ Write equations of ellipses in standard form and graph ellipses *(p. 744)*.	21–24
☐ Use properties of ellipses to model and solve real-life problems *(p. 748)*.	25, 26
☐ Find the eccentricities of ellipses *(p. 748)*.	27–30
Section 10.4	
☐ Write equations of hyperbolas in standard form *(p. 753)*.	31–34
☐ Find asymptotes of and graph hyperbolas *(p. 755)*.	35–38
☐ Use properties of hyperbolas to solve real-life problems *(p. 758)*.	39, 40
☐ Classify conics from their general equations *(p. 759)*.	41–44
Section 10.5	
☐ Rotate the coordinate axes to eliminate the *xy*-term in equations of conics *(p. 763)*.	45–48
☐ Use the discriminant to classify conics *(p. 767)*.	49–52
Section 10.6	
☐ Evaluate sets of parametric equations for given values of the parameter *(p. 771)*.	53, 54
☐ Sketch curves that are represented by sets of parametric equations *(p. 772)*. and rewrite the equations as single rectangular equations *(p. 773)*.	55–60
☐ Find sets of parametric equations for graphs *(p. 774)*.	61–64
Section 10.7	
☐ Plot points on the polar coordinate system *(p. 779)*.	65–68
☐ Convert points from rectangular to polar form and vice versa *(p. 780)*.	69–76
☐ Convert equations from rectangular to polar form and vice versa *(p. 782)*.	77–88
Section 10.8	
☐ Graph polar equations by point plotting *(p. 785)*.	89–98
☐ Use symmetry *(p. 786)*, zeros, and maximum *r*-values *(p. 787)* to sketch graphs of polar equations.	89–98
☐ Recognize special polar graphs *(p. 789)*.	99–102
Section 10.9	
☐ Define conics in terms of eccentricity and write and graph equations of conics in polar form *(p. 793)*.	103–110
☐ Use equations of conics in polar form to model real-life problems *(p. 796)*.	111, 112

10 Review Exercises

10.1 In Exercises 1–4, find the inclination θ (in radians and degrees) of the line with the given characteristics.

1. Passes through the points $(-1, 2)$ and $(2, 5)$

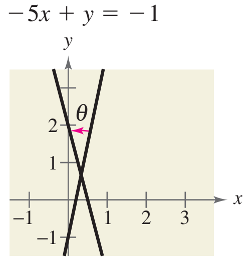
2. Passes through the points $(3, 4)$ and $(-2, 7)$

3. Equation: $y = 2x + 4$

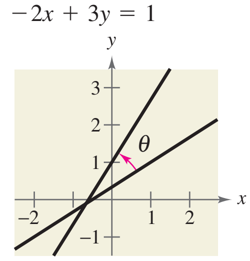
4. Equation: $6x - 7y - 5 = 0$

In Exercises 5–8, find the angle θ (in radians and degrees) between the lines.

5. $4x + y = 2$
$-5x + y = -1$



6. $-5x + 3y = 3$
$-2x + 3y = 1$



7. $2x - 7y = 8$
$0.4x + y = 0$

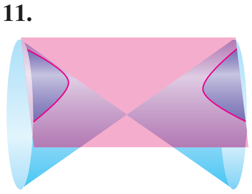
8. $0.02x + 0.07y = 0.18$
$0.09x - 0.04y = 0.17$

In Exercises 9 and 10, find the distance between the point and the line.

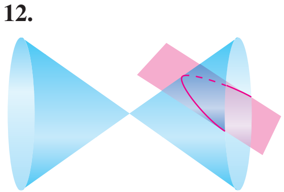
	Point	*Line*
9.	$(1, 2)$	$x - y - 3 = 0$
10.	$(0, 4)$	$x + 2y - 2 = 0$

10.2 In Exercises 11 and 12, state what type of conic is formed by the intersection of the plane and the double-napped cone.

11.



12.



In Exercises 13–16, find the standard form of the equation of the parabola with the given characteristics. Then graph the parabola.

13. Vertex: $(0, 0)$
Focus: $(4, 0)$

14. Vertex: $(2, 0)$
Focus: $(0, 0)$

15. Vertex: $(0, 2)$
Directrix: $x = -3$

16. Vertex: $(2, 2)$
Directrix: $y = 0$

In Exercises 17 and 18, find an equation of the tangent line to the parabola at the given point, and find the x-intercept of the line.

17. $x^2 = -2y,\ (2, -2)$

18. $x^2 = -2y,\ (-4, -8)$

19. ***Architecture*** A parabolic archway is 12 meters high at the vertex. At a height of 10 meters, the width of the archway is 8 meters (see figure). How wide is the archway at ground level?

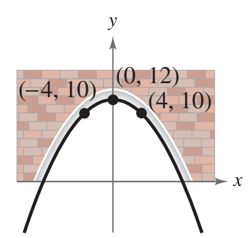


FIGURE FOR 19

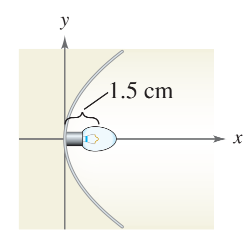


FIGURE FOR 20

20. ***Flashlight*** The light bulb in a flashlight is at the focus of its parabolic reflector, 1.5 centimeters from the vertex of the reflector (see figure). Write an equation of a cross section of the flashlight's reflector with its focus on the positive x-axis and its vertex at the origin.

10.3 In Exercises 21–24, find the standard form of the equation of the ellipse with the given characteristics. Then graph the ellipse.

21. Vertices: $(-3, 0), (7, 0)$; foci: $(0, 0), (4, 0)$

22. Vertices: $(2, 0), (2, 4)$; foci: $(2, 1), (2, 3)$

23. Vertices: $(0, 1), (4, 1)$; endpoints of the minor axis: $(2, 0), (2, 2)$

24. Vertices: $(-4, -1), (-4, 11)$; endpoints of the minor axis: $(-6, 5), (-2, 5)$

25. ***Architecture*** A semielliptical archway is to be formed over the entrance to an estate. The arch is to be set on pillars that are 10 feet apart and is to have a height (atop the pillars) of 4 feet. Where should the foci be placed in order to sketch the arch?

26. ***Wading Pool*** You are building a wading pool that is in the shape of an ellipse. Your plans give an equation for the elliptical shape of the pool measured in feet as

$$\frac{x^2}{324} + \frac{y^2}{196} = 1.$$

Find the longest distance across the pool, the shortest distance, and the distance between the foci.

In Exercises 27–30, find the center, vertices, foci, and eccentricity of the ellipse.

27. $\dfrac{(x+2)^2}{81} + \dfrac{(y-1)^2}{100} = 1$

28. $\dfrac{(x-5)^2}{1} + \dfrac{(y+3)^2}{36} = 1$

29. $16x^2 + 9y^2 - 32x + 72y + 16 = 0$

30. $4x^2 + 25y^2 + 16x - 150y + 141 = 0$

10.4 In Exercises 31–34, find the standard form of the equation of the hyperbola with the given characteristics.

31. Vertices: $(0, \pm 1)$; foci: $(0, \pm 3)$

32. Vertices: $(2, 2), (-2, 2)$; foci: $(4, 2), (-4, 2)$

33. Foci: $(0, 0), (8, 0)$; asymptotes: $y = \pm 2(x - 4)$

34. Foci: $(3, \pm 2)$; asymptotes: $y = \pm 2(x - 3)$

In Exercises 35–38, find the center, vertices, foci, and the equations of the asymptotes of the hyperbola, and sketch its graph using the asymptotes as an aid.

35. $\dfrac{(x-3)^2}{16} - \dfrac{(y+5)^2}{4} = 1$

36. $\dfrac{(y-1)^2}{4} - x^2 = 1$

37. $9x^2 - 16y^2 - 18x - 32y - 151 = 0$

38. $-4x^2 + 25y^2 - 8x + 150y + 121 = 0$

39. ***LORAN*** Radio transmitting station A is located 200 miles east of transmitting station B. A ship is in an area to the north and 40 miles west of station A. Synchronized radio pulses transmitted at 186,000 miles per second by the two stations are received 0.0005 second sooner from station A than from station B. How far north is the ship?

40. ***Locating an Explosion*** Two of your friends live 4 miles apart and on the same "east-west" street, and you live halfway between them. You are having a three-way phone conversation when you hear an explosion. Six seconds later, your friend to the east hears the explosion, and your friend to the west hears it 8 seconds after you do. Find equations of two hyperbolas that would locate the explosion. (Assume that the coordinate system is measured in feet and that sound travels at 1100 feet per second.)

In Exercises 41–44, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

41. $5x^2 - 2y^2 + 10x - 4y + 17 = 0$

42. $-4y^2 + 5x + 3y + 7 = 0$

43. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

44. $4x^2 + 4y^2 - 4x + 8y - 11 = 0$

10.5 In Exercises 45–48, rotate the axes to eliminate the *xy*-term in the equation. Then write the equation in standard form. Sketch the graph of the resulting equation, showing both sets of axes.

45. $xy - 4 = 0$

46. $x^2 - 10xy + y^2 + 1 = 0$

47. $5x^2 - 2xy + 5y^2 - 12 = 0$

48. $4x^2 + 8xy + 4y^2 + 7\sqrt{2}x + 9\sqrt{2}y = 0$

In Exercises 49–52, (a) use the discriminant to classify the graph, (b) use the Quadratic Formula to solve for *y*, and (c) use a graphing utility to graph the equation.

49. $16x^2 - 24xy + 9y^2 - 30x - 40y = 0$

50. $13x^2 - 8xy + 7y^2 - 45 = 0$

51. $x^2 + y^2 + 2xy + 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0$

52. $x^2 - 10xy + y^2 + 1 = 0$

10.6 In Exercises 53 and 54, complete the table for each set of parametric equations. Plot the points (x, y) and sketch a graph of the parametric equations.

53. $x = 3t - 2$ and $y = 7 - 4t$

t	-3	-2	-1	0	1	2	3
x							
y							

54. $x = \dfrac{1}{5}t$ and $y = \dfrac{4}{t-1}$

t	-1	0	2	3	4	5
x						
y						

In Exercises 55–60, (a) sketch the curve represented by the parametric equations (indicate the orientation of the curve) and (b) eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve. Adjust the domain of the resulting rectangular equation, if necessary. (c) Verify your result with a graphing utility.

55. $x = 2t$
$y = 4t$

56. $x = 1 + 4t$
$y = 2 - 3t$

57. $x = t^2$
$y = \sqrt{t}$

58. $x = t + 4$
$y = t^2$

59. $x = 6\cos\theta$
$y = 6\sin\theta$

60. $x = 3 + 3\cos\theta$
$y = 2 + 5\sin\theta$

61. Find a parametric representation of the circle with center $(5, 4)$ and radius 6.

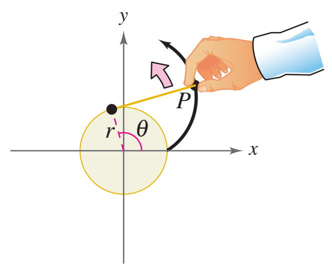
62. Find a parametric representation of the ellipse with center $(-3, 4)$, major axis horizontal and eight units in length, and minor axis six units in length.

63. Find a parametric representation of the hyperbola with vertices $(0, \pm 4)$ and foci $(0, \pm 5)$.

64. ***Involute of a Circle*** The *involute* of a circle is described by the endpoint P of a string that is held taut as it is unwound from a spool (see figure). The spool does not rotate. Show that a parametric representation of the involute of a circle is

$$x = r(\cos\theta + \theta\sin\theta)$$

$$y = r(\sin\theta - \theta\cos\theta).$$



10.7 **In Exercises 65–68, plot the point given in polar coordinates and find two additional polar representations of the point, using $-2\pi < \theta < 2\pi$.**

65. $\left(2, \frac{\pi}{4}\right)$

66. $\left(-5, -\frac{\pi}{3}\right)$

67. $(-7, 4.19)$

68. $(\sqrt{3}, 2.62)$

In Exercises 69–72, a point in polar coordinates is given. Convert the point to rectangular coordinates.

69. $\left(-1, \frac{\pi}{3}\right)$

70. $\left(2, \frac{5\pi}{4}\right)$

71. $\left(3, \frac{3\pi}{4}\right)$

72. $\left(0, \frac{\pi}{2}\right)$

In Exercises 73–76, a point in rectangular coordinates is given. Convert the point to polar coordinates.

73. $(0, 2)$

74. $(-\sqrt{5}, \sqrt{5})$

75. $(4, 6)$

76. $(3, -4)$

In Exercises 77–82, convert the rectangular equation to polar form.

77. $x^2 + y^2 = 49$

78. $x^2 + y^2 = 20$

79. $x^2 + y^2 - 6y = 0$

80. $x^2 + y^2 - 4x = 0$

81. $xy = 5$

82. $xy = -2$

In Exercises 83–88, convert the polar equation to rectangular form.

83. $r = 5$

84. $r = 12$

85. $r = 3\cos\theta$

86. $r = 8\sin\theta$

87. $r^2 = \sin\theta$

88. $r^2 = \cos 2\theta$

10.8 **In Exercises 89–98, determine the symmetry of r, the maximum value of $|r|$, and any zeros of r. Then sketch the graph of the polar equation (plot additional points if necessary).**

89. $r = 4$

90. $r = 11$

91. $r = 4\sin 2\theta$

92. $r = \cos 5\theta$

93. $r = -2(1 + \cos\theta)$

94. $r = 3 - 4\cos\theta$

95. $r = 2 + 6\sin\theta$

96. $r = 5 - 5\cos\theta$

97. $r = -3\cos 2\theta$

98. $r = \cos 2\theta$

In Exercises 99–102, identify the type of polar graph and use a graphing utility to graph the equation.

99. $r = 3(2 - \cos\theta)$

100. $r = 3(1 - 2\cos\theta)$

101. $r = 4\cos 3\theta$

102. $r^2 = 9\cos 2\theta$

10.9 **In Exercises 103–106, identify the conic and sketch its graph.**

103. $r = \dfrac{1}{1 + 2\sin\theta}$

104. $r = \dfrac{2}{1 + \sin\theta}$

105. $r = \dfrac{4}{5 - 3\cos\theta}$

106. $r = \dfrac{16}{4 + 5\cos\theta}$

In Exercises 107–110, find a polar equation of the conic with its focus at the pole.

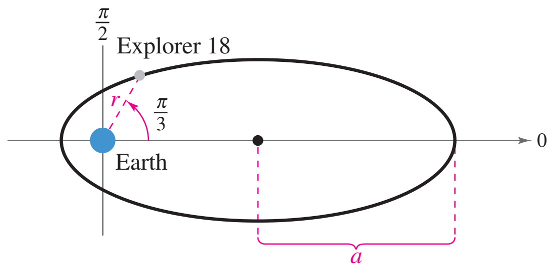
107. Parabola Vertex: $(2, \pi)$

108. Parabola Vertex: $(2, \pi/2)$

109. Ellipse Vertices: $(5, 0), (1, \pi)$

110. Hyperbola Vertices: $(1, 0), (7, 0)$

111. ***Explorer 18*** On November 26, 1963, the United States launched Explorer 18. Its low and high points above the surface of Earth were 119 miles and 122,800 miles, respectively (see figure). The center of Earth was at one focus of the orbit. Find the polar equation of the orbit and find the distance between the surface of Earth (assume Earth has a radius of 4000 miles) and the satellite when $\theta = \pi/3$.



112. ***Asteroid*** An asteroid takes a parabolic path with Earth as its focus. It is about 6,000,000 miles from Earth at its closest approach. Write the polar equation of the path of the asteroid with its vertex at $\theta = \pi/2$. Find the distance between the asteroid and Earth when $\theta = -\pi/3$.

Synthesis

True or False? **In Exercises 113–116, determine whether the statement is true or false. Justify your answer.**

113. When $B = 0$ in an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the graph of the equation can be a parabola only if $C = 0$ also.

114. The graph of $\frac{1}{4}x^2 - y^4 = 1$ is a hyperbola.

115. Only one set of parametric equations can represent the line $y = 3 - 2x$.

116. There is a unique polar coordinate representation of each point in the plane.

117. Consider an ellipse with the major axis horizontal and 10 units in length. The number b in the standard form of the equation of the ellipse must be less than what real number? Explain the change in the shape of the ellipse as b approaches this number.

118. The graph of the parametric equations $x = 2 \sec t$ and $y = 3 \tan t$ is shown in the figure. How would the graph change for the equations $x = 2 \sec(-t)$ and $y = 3 \tan(-t)$?

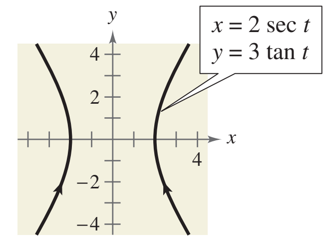
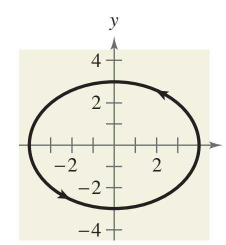


FIGURE FOR 118

119. A moving object is modeled by the parametric equations $x = 4 \cos t$ and $y = 3 \sin t$, where t is time (see figure). How would the path change for the following?

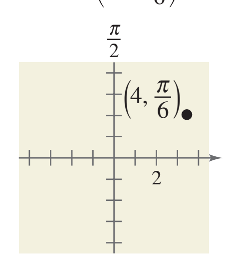
(a) $x = 4 \cos 2t, \quad y = 3 \sin 2t$

(b) $x = 5 \cos t, \quad y = 3 \sin t$



120. Identify the type of symmetry each of the following polar points has with the point in the figure.

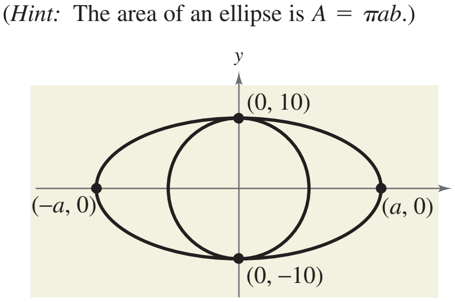
(a) $\left(-4, \frac{\pi}{6}\right)$ (b) $\left(4, -\frac{\pi}{6}\right)$ (c) $\left(-4, -\frac{\pi}{6}\right)$



121. What is the relationship between the graphs of the rectangular and polar equations?

(a) $x^2 + y^2 = 25, \quad r = 5$ (b) $x - y = 0, \quad \theta = \frac{\pi}{4}$

122. ***Geometry*** The area of the ellipse in the figure is twice the area of the circle. What is the length of the major axis? (*Hint:* The area of an ellipse is $A = \pi ab$.)



10 Chapter Test

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Find the inclination of the line $2x - 7y + 3 = 0$.

2. Find the angle between the lines $3x + 2y - 4 = 0$ and $4x - y + 6 = 0$.

3. Find the distance between the point $(7, 5)$ and the line $y = 5 - x$.

In Exercises 4–7, classify the conic and write the equation in standard form. Identify the center, vertices, foci, and asymptotes (if applicable). Then sketch the graph of the conic.

4. $y^2 - 4x + 4 = 0$

5. $x^2 - 4y^2 - 4x = 0$

6. $9x^2 + 16y^2 + 54x - 32y - 47 = 0$

7. $2x^2 + 2y^2 - 8x - 4y + 9 = 0$

8. Find the standard form of the equation of the parabola with vertex $(3, -2)$, with a vertical axis, and passing through the point $(0, 4)$.

9. Find the standard form of the equation of the hyperbola with foci $(0, 0)$ and $(0, 4)$ and asymptotes $y = \pm\frac{1}{2}x + 2$.

10. (a) Determine the number of degrees the axis must be rotated to eliminate the xy-term of the conic $x^2 + 6xy + y^2 - 6 = 0$.

(b) Graph the conic from part (a) and use a graphing utility to confirm your result.

11. Sketch the curve represented by the parametric equations $x = 2 + 3\cos\theta$ and $y = 2\sin\theta$. Eliminate the parameter and write the corresponding rectangular equation.

12. Find a set of parametric equations of the line passing through the points $(2, -3)$ and $(6, 4)$. (There are many correct answers.)

13. Convert the polar coordinate $\left(-2, \frac{5\pi}{6}\right)$ to rectangular form.

14. Convert the rectangular coordinate $(2, -2)$ to polar form and find two additional polar representations of this point.

15. Convert the rectangular equation $x^2 + y^2 - 4y = 0$ to polar form.

In Exercises 16–19, sketch the graph of the polar equation. Identify the type of graph.

16. $r = \dfrac{4}{1 + \cos\theta}$

17. $r = \dfrac{4}{2 + \cos\theta}$

18. $r = 2 + 3\sin\theta$

19. $r = 3\sin 2\theta$

20. Find a polar equation of the ellipse with focus at the pole, eccentricity $e = \frac{1}{4}$, and directrix $y = 4$.

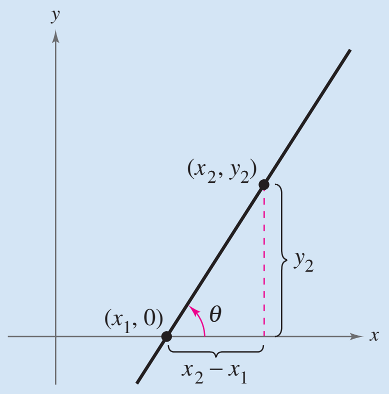
21. A straight road rises with an inclination of 0.15 radian from the horizontal. Find the slope of the road and the change in elevation over a one-mile stretch of the road.

22. A baseball is hit at a point 3 feet above the ground toward the left field fence. The fence is 10 feet high and 375 feet from home plate. The path of the baseball can be modeled by the parametric equations $x = (115\cos\theta)t$ and $y = 3 + (115\sin\theta)t - 16t^2$. Will the baseball go over the fence if it is hit at an angle of $\theta = 30°$? Will the baseball go over the fence if $\theta = 35°$?

Proofs in Mathematics

Inclination and Slope (p. 728)

If a nonvertical line has inclination θ and slope m, then $m = \tan\theta$.



Proof

If $m = 0$, the line is horizontal and $\theta = 0$. So, the result is true for horizontal lines because $m = 0 = \tan 0$.

If the line has a positive slope, it will intersect the x-axis. Label this point $(x_1, 0)$, as shown in the figure. If (x_2, y_2) is a second point on the line, the slope is

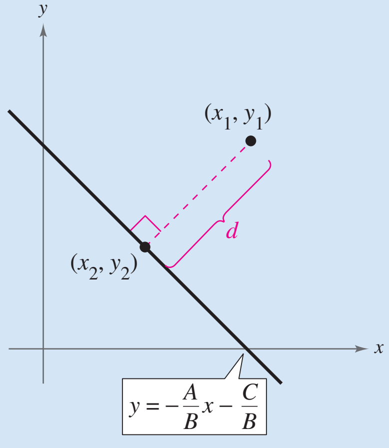
$$m = \frac{y_2 - 0}{x_2 - x_1} = \frac{y_2}{x_2 - x_1} = \tan\theta.$$

The case in which the line has a negative slope can be proved in a similar manner.

Distance Between a Point and a Line (p. 730)

The distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$



Proof

For simplicity's sake, assume that the given line is neither horizontal nor vertical (see figure). By writing the equation $Ax + By + C = 0$ in slope-intercept form

$$y = -\frac{A}{B}x - \frac{C}{B}$$

you can see that the line has a slope of $m = -A/B$. So, the slope of the line passing through (x_1, y_1) and perpendicular to the given line is B/A, and its equation is $y - y_1 = (B/A)(x - x_1)$. These two lines intersect at the point (x_2, y_2), where

$$x_2 = \frac{B(Bx_1 - Ay_1) - AC}{A^2 + B^2} \quad \text{and} \quad y_2 = \frac{A(-Bx_1 + Ay_1) - BC}{A^2 + B^2}.$$

Finally, the distance between (x_1, y_1) and (x_2, y_2) is

$$\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{\left(\frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2} - x_1\right)^2 + \left(\frac{-ABx_1 + A^2y_1 - BC}{A^2 + B^2} - y_1\right)^2} \\
&= \sqrt{\frac{A^2(Ax_1 + By_1 + C)^2 + B^2(Ax_1 + By_1 + C)^2}{(A^2 + B^2)^2}} \\
&= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.
\end{aligned}$$

Parabolic Paths

There are many natural occurrences of parabolas in real life. For instance, the famous astronomer Galileo discovered in the 17th century that an object that is projected upward and obliquely to the pull of gravity travels in a parabolic path. Examples of this are the center of gravity of a jumping dolphin and the path of water molecules in a drinking fountain.

Standard Equation of a Parabola *(p. 736)*

The standard form of the equation of a parabola with vertex at (h, k) is as follows.

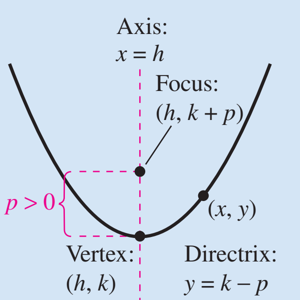
$(x - h)^2 = 4p(y - k), \quad p \neq 0$ Vertical axis, directrix: $y = k - p$

$(y - k)^2 = 4p(x - h), \quad p \neq 0$ Horizontal axis, directrix: $x = h - p$

The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin $(0, 0)$, the equation takes one of the following forms.

$x^2 = 4py$ Vertical axis

$y^2 = 4px$ Horizontal axis



Parabola with vertical axis

Proof

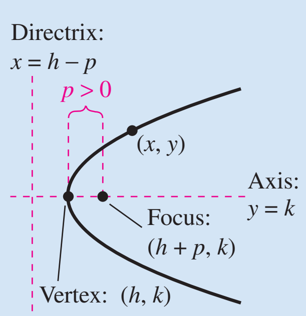
For the case in which the directrix is parallel to the x-axis and the focus lies above the vertex, as shown in the top figure, if (x, y) is any point on the parabola, then, by definition, it is equidistant from the focus $(h, k + p)$ and the directrix $y = k - p$. So, you have

$$\sqrt{(x - h)^2 + [y - (k + p)]^2} = y - (k - p)$$
$$(x - h)^2 + [y - (k + p)]^2 = [y - (k - p)]^2$$
$$(x - h)^2 + y^2 - 2y(k + p) + (k + p)^2 = y^2 - 2y(k - p) + (k - p)^2$$
$$(x - h)^2 + y^2 - 2ky - 2py + k^2 + 2pk + p^2 = y^2 - 2ky + 2py + k^2 - 2pk + p^2$$
$$(x - h)^2 - 2py + 2pk = 2py - 2pk$$
$$(x - h)^2 = 4p(y - k).$$

For the case in which the directrix is parallel to the y-axis and the focus lies to the right of the vertex, as shown in the bottom figure, if (x, y) is any point on the parabola, then, by definition, it is equidistant from the focus $(h + p, k)$ and the directrix $x = h - p$. So, you have

$$\sqrt{[x - (h + p)]^2 + (y - k)^2} = x - (h - p)$$
$$[x - (h + p)]^2 + (y - k)^2 = [x - (h - p)]^2$$
$$x^2 - 2x(h + p) + (h + p)^2 + (y - k)^2 = x^2 - 2x(h - p) + (h - p)^2$$
$$x^2 - 2hx - 2px + h^2 + 2ph + p^2 + (y - k)^2 = x^2 - 2hx + 2px + h^2 - 2ph + p^2$$
$$-2px + 2ph + (y - k)^2 = 2px - 2ph$$
$$(y - k)^2 = 4p(x - h).$$

Note that if a parabola is centered at the origin, then the two equations above would simplify to $x^2 = 4py$ and $y^2 = 4px$, respectively.



Parabola with horizontal axis

Polar Equations of Conics *(p. 793)*

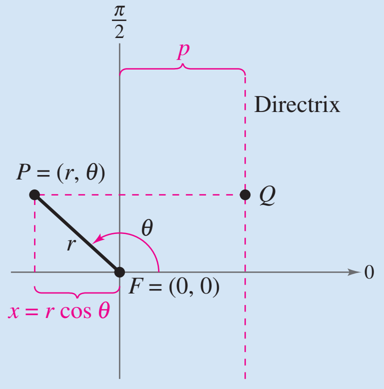
The graph of a polar equation of the form

1. $r = \dfrac{ep}{1 \pm e \cos \theta}$

or

2. $r = \dfrac{ep}{1 \pm e \sin \theta}$

is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.



Proof

A proof for $r = ep/(1 + e \cos \theta)$ with $p > 0$ is shown here. The proofs of the other cases are similar. In the figure, consider a vertical directrix, p units to the right of the focus $F = (0, 0)$. If $P = (r, \theta)$ is a point on the graph of

$$r = \frac{ep}{1 + e \cos \theta}$$

the distance between P and the directrix is

$$\begin{aligned} PQ &= |p - x| \\ &= |p - r \cos \theta| \\ &= \left| p - \left(\frac{ep}{1 + e \cos \theta} \right) \cos \theta \right| \\ &= \left| p \left(1 - \frac{e \cos \theta}{1 + e \cos \theta} \right) \right| \\ &= \left| \frac{p}{1 + e \cos \theta} \right| \\ &= \left| \frac{r}{e} \right|. \end{aligned}$$

Moreover, because the distance between P and the pole is simply $PF = |r|$, the ratio of PF to PQ is

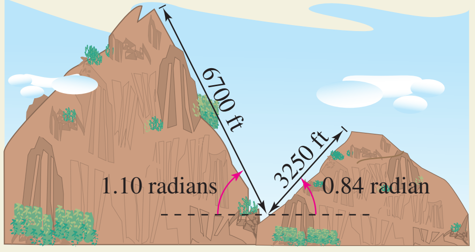
$$\begin{aligned} \frac{PF}{PQ} &= \frac{|r|}{\left| \frac{r}{e} \right|} \\ &= |e| \\ &= e \end{aligned}$$

and, by definition, the graph of the equation must be a conic.

P.S. Problem Solving

This collection of thought-provoking and challenging exercises further explores and expands upon concepts learned in this chapter.

1. Several mountain climbers are located in a mountain pass between two peaks. The angles of elevation to the two peaks are 0.84 radian and 1.10 radians. A range finder shows that the distances to the peaks are 3250 feet and 6700 feet, respectively (see figure).



(a) Find the angle between the two lines of sight to the peaks.

(b) Approximate the amount of vertical climb that is necessary to reach the summit of each peak.

2. Statuary Hall is an elliptical room in the United States Capitol in Washington D.C. The room is also called the Whispering Gallery because a person standing at one focus of the room can hear even a whisper spoken by a person standing at the other focus. This occurs because any sound that is emitted from one focus of an ellipse will reflect off the side of the ellipse to the other focus. Statuary Hall is 46 feet wide and 97 feet long.

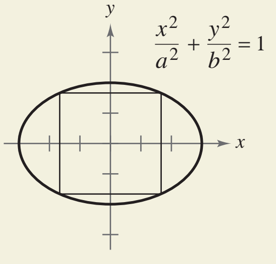
(a) Find an equation that models the shape of the room.

(b) How far apart are the two foci?

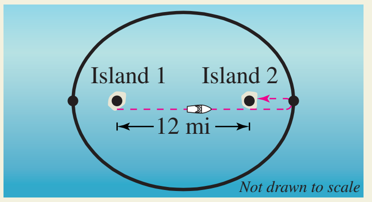
(c) What is the area of the floor of the room? (The area of an ellipse is $A = \pi ab$.)

3. Find the equation(s) of all parabolas that have the x-axis as the axis of symmetry and focus at the origin.

4. Find the area of the square inscribed in the ellipse below.



5. A tour boat travels between two islands that are 12 miles apart (see figure). For a trip between the islands, there is enough fuel for a 20-mile trip.



(a) Explain why the region in which the boat can travel is bounded by an ellipse.

(b) Let $(0, 0)$ represent the center of the ellipse. Find the coordinates of each island.

(c) The boat travels from one island, straight past the other island to the vertex of the ellipse, and back to the second island. How many miles does the boat travel? Use your answer to find the coordinates of the vertex.

(d) Use the results from parts (b) and (c) to write an equation for the ellipse that bounds the region in which the boat can travel.

6. Find an equation of the hyperbola such that for any point on the hyperbola, the difference between its distances from the points $(2, 2)$ and $(10, 2)$ is 6.

7. Prove that the graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is one of the following (except in degenerate cases).

Conic	*Condition*
(a) Circle	$A = C$
(b) Parabola	$A = 0$ or $C = 0$ (but not both)
(c) Ellipse	$AC > 0$
(d) Hyperbola	$AC < 0$

8. The following sets of parametric equations model projectile motion.

$$x = (v_0 \cos \theta)t \qquad x = (v_0 \cos \theta)t$$

$$y = (v_0 \sin \theta)t \qquad y = h + (v_0 \sin \theta)t - 16t^2$$

(a) Under what circumstances would you use each model?

(b) Eliminate the parameter for each set of equations.

(c) In which case is the path of the moving object not affected by a change in the velocity v? Explain.

9. As t increases, the ellipse given by the parametric equations

$$x = \cos t \text{ and } y = 2 \sin t$$

is traced out *counterclockwise*. Find a parametric representation for which the same ellipse is traced out *clockwise*.

10. A **hypocycloid** has the parametric equations

$$x = (a - b)\cos t + b \cos\left(\frac{a-b}{b}t\right)$$

and

$$y = (a - b)\sin t - b \sin\left(\frac{a-b}{b}t\right).$$

Use a graphing utility to graph the hypocycloid for each value of a and b. Describe each graph.

(a) $a = 2, b = 1$ (b) $a = 3, b = 1$

(c) $a = 4, b = 1$ (d) $a = 10, b = 1$

(e) $a = 3, b = 2$ (f) $a = 4, b = 3$

11. The curve given by the parametric equations

$$x = \frac{1 - t^2}{1 + t^2}$$

and

$$y = \frac{t(1 - t^2)}{1 + t^2}$$

is called a **strophoid.**

(a) Find a rectangular equation of the strophoid.

(b) Find a polar equation of the strophoid.

(c) Use a graphing utility to graph the strophoid.

12. The rose curves described in this chapter are of the form

$$r = a \cos n\theta \quad \text{or} \quad r = a \sin n\theta$$

where n is a positive integer that is greater than or equal to 2. Use a graphing utility to graph $r = a \cos n\theta$ and $r = a \sin n\theta$ for some noninteger values of n. Describe the graphs.

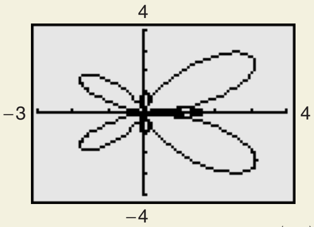
13. What conic section is represented by the polar equation

$$r = a \sin\theta + b \cos\theta?$$

14. The graph of the polar equation

$$r = e^{\cos\theta} - 2\cos 4\theta + \sin^5\left(\frac{\theta}{12}\right)$$

is called the *butterfly curve*, as shown in the figure.



$r = e^{\cos\theta} - 2\cos 4\theta + \sin^5\left(\frac{\theta}{12}\right)$

FIGURE FOR 14

(a) The graph above was produced using $0 \le \theta \le 2\pi$. Does this show the entire graph? Explain your reasoning.

(b) Approximate the maximum r-value of the graph. Does this value change if you use $0 \le \theta \le 4\pi$ instead of $0 \le \theta \le 2\pi$? Explain.

15. Use a graphing utility to graph the polar equation

$$r = \cos 5\theta + n \cos\theta$$

for $0 \le \theta \le \pi$ for the integers $n = -5$ to $n = 5$. As you graph these equations, you should see the graph change shape from a heart to a bell. Write a short paragraph explaining what values of n produce the heart portion of the curve and what values of n produce the bell portion.

16. The planets travel in elliptical orbits with the sun at one focus. The polar equation of the orbit of a planet with one focus at the pole and major axis of length $2a$ is

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta}$$

where e is the eccentricity. The minimum distance (perihelion) from the sun to a planet is $r = a(1 - e)$ and the maximum distance (aphelion) is $r = a(1 + e)$. The length of the major axis for the planet Neptune is $a = 9.000 \times 10^9$ kilometers and the eccentricity is $e = 0.0086$. The length of the major axis for the planet Pluto is $a = 10.813 \times 10^9$ kilometers and the eccentricity is $e = 0.2488$.

(a) Find the polar equation of the orbit of each planet.

(b) Find the perihelion and aphelion distances for each planet.

(c) Use a graphing utility to graph the polar equation of each planet's orbit in the same viewing window.

(d) Do the orbits of the two planets intersect? Will the two planets ever collide? Why or why not?

(e) Is Pluto ever closer to the sun than Neptune? Why is Pluto called the ninth planet and Neptune the eighth planet?