10.8 Graphs of Polar Equations

What you should learn

- Graph polar equations by point plotting.
- Use symmetry to sketch graphs of polar equations.
- Use zeros and maximum r-values to sketch graphs of polar equations.
- Recognize special polar graphs.

Why you should learn it

Equations of several common figures are simpler in polar form than in rectangular form. For instance, Exercise 6 on page 791 shows the graph of a circle and its polar equation.

Emphasize setting up the table of θ values. Your students will benefit from labeling the points as they plot them.

Introduction

In previous chapters, you spent a lot of time learning how to sketch graphs on rectangular coordinate systems. You began with the basic point-plotting method, which was then enhanced by sketching aids such as symmetry, intercepts, asymptotes, periods, and shifts. This section approaches curve sketching on the polar coordinate system similarly, beginning with a demonstration of point plotting.

Example 1 **Graphing a Polar Equation by Point Plotting**

Sketch the graph of the polar equation $r = 4 \sin \theta$.

Solution

The sine function is periodic, so you can get a full range of r-values by considering values of θ in the interval $0 \le \theta \le 2\pi$, as shown in the following table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	0	2	$2\sqrt{3}$	4	$2\sqrt{3}$	2	0	-2	-4	-2	0

If you plot these points as shown in Figure 10.69, it appears that the graph is a circle of radius 2 whose center is at the point $(x, y) = (0, 2)$.

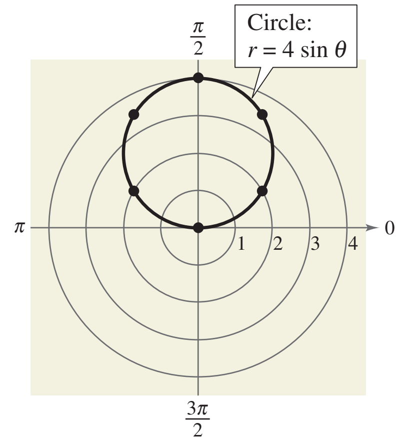


FIGURE 10.69

CHECKPOINT Now try Exercise 21.

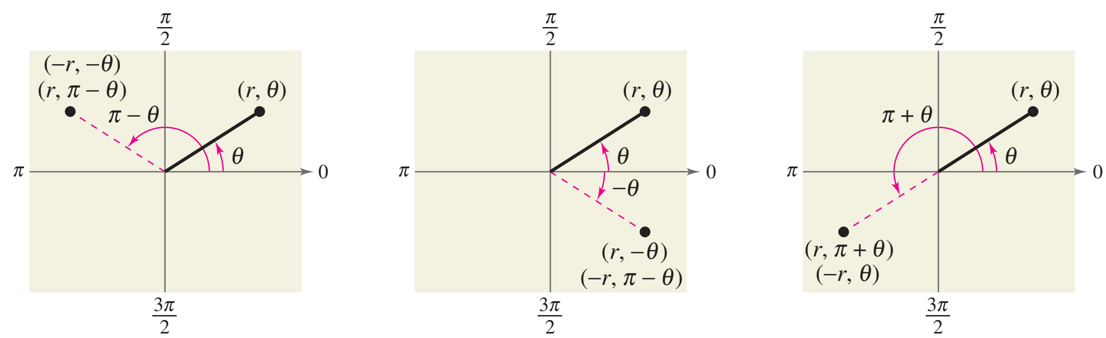
You can confirm the graph in Figure 10.69 by converting the polar equation to rectangular form and then sketching the graph of the rectangular equation. You can also use a graphing utility set to *polar* mode and graph the polar equation or set the graphing utility to *parametric* mode and graph a parametric representation.

Symmetry

Point out to your students that these tests are sufficient for showing symmetry; however, they are not necessary. A polar graph can exhibit symmetry even when the tests fail to indicate symmetry.

In Figure 10.69, note that as θ increases from 0 to 2π the graph is traced out twice. Moreover, note that the graph is *symmetric with respect to the line* $\theta = \pi/2$. Had you known about this symmetry and retracing ahead of time, you could have used fewer points.

Symmetry with respect to the line $\theta = \pi/2$ is one of three important types of symmetry to consider in polar curve sketching. (See Figure 10.70.)



Symmetry with Respect to the Line $\theta = \dfrac{\pi}{2}$ *Symmetry with Respect to the Polar Axis* *Symmetry with Respect to the Pole*

FIGURE 10.70

STUDY TIP

Note in Example 2 that $\cos(-\theta) = \cos\theta$. This is because the cosine function is *even*. Recall from Section 4.2 that the cosine function is even and the sine function is odd. That is, $\sin(-\theta) = -\sin\theta$.

Tests for Symmetry in Polar Coordinates

The graph of a polar equation is symmetric with respect to the following if the given substitution yields an equivalent equation.

1. *The line* $\theta = \pi/2$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$.
2. *The polar axis:* Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$.
3. *The pole:* Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$.

Example 2 Using Symmetry to Sketch a Polar Graph

Use symmetry to sketch the graph of $r = 3 + 2\cos\theta$.

Solution

Replacing (r, θ) by $(r, -\theta)$ produces $r = 3 + 2\cos(-\theta) = 3 + 2\cos\theta$. So, you can conclude that the curve is symmetric with respect to the polar axis. Plotting the points in the table and using polar axis symmetry, you obtain the graph shown in Figure 10.71. This graph is called a **limaçon.**

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	5	4	3	2	1

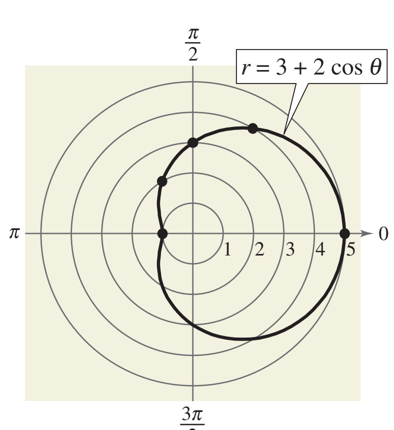


FIGURE 10.71

CHECKPOINT Now try Exercise 27.

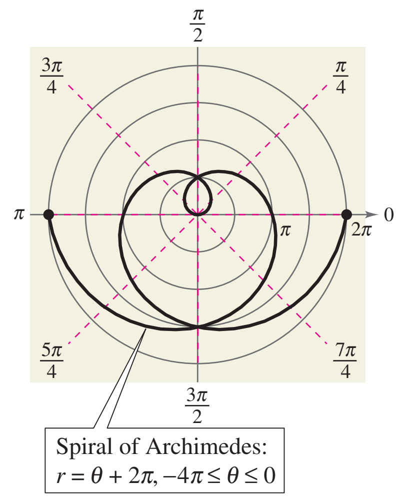


FIGURE 10.72

The three tests for symmetry in polar coordinates listed on page 786 are sufficient to guarantee symmetry, but they are not necessary. For instance, Figure 10.72 shows the graph of $r = \theta + 2\pi$ to be symmetric with respect to the line $\theta = \pi/2$, and yet the tests on page 786 fail to indicate symmetry because neither of the following replacements yields an equivalent equation.

Original Equation	*Replacement*	*New Equation*
$r = \theta + 2\pi$	(r, θ) by $(-r, -\theta)$	$-r = -\theta + 2\pi$
$r = \theta + 2\pi$	(r, θ) by $(r, \pi - \theta)$	$r = -\theta + 3\pi$

The equations discussed in Examples 1 and 2 are of the form

$r = 4 \sin \theta = f(\sin \theta)$ and $r = 3 + 2 \cos \theta = g(\cos \theta)$.

The graph of the first equation is symmetric with respect to the line $\theta = \pi/2$, and the graph of the second equation is symmetric with respect to the polar axis. This observation can be generalized to yield the following tests.

Quick Tests for Symmetry in Polar Coordinates

1. The graph of $r = f(\sin \theta)$ is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

2. The graph of $r = g(\cos \theta)$ is symmetric with respect to the polar axis.

Zeros and Maximum *r*-Values

Two additional aids to graphing of polar equations involve knowing the θ-values for which $|r|$ is maximum and knowing the θ-values for which $r = 0$. For instance, in Example 1, the maximum value of $|r|$ for $r = 4 \sin \theta$ is $|r| = 4$, and this occurs when $\theta = \pi/2$, as shown in Figure 10.69. Moreover, $r = 0$ when $\theta = 0$.

Example 3 Sketching a Polar Graph

Sketch the graph of $r = 1 - 2 \cos \theta$.

Solution

From the equation $r = 1 - 2 \cos \theta$, you can obtain the following.

Symmetry: With respect to the polar axis

Maximum value of $|r|$*:* $r = 3$ when $\theta = \pi$

Zero of r: $r = 0$ when $\theta = \pi/3$

The table shows several θ-values in the interval $[0, \pi]$. By plotting the corresponding points, you can sketch the graph shown in Figure 10.73.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	-1	-0.73	0	1	2	2.73	3

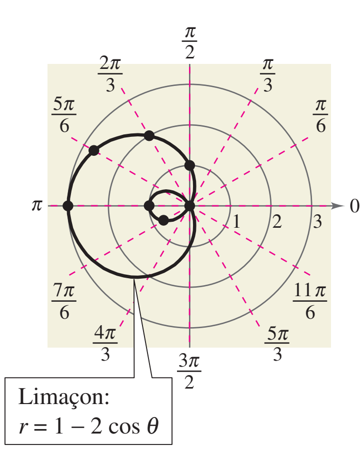


FIGURE 10.73

Note how the negative r-values determine the *inner loop* of the graph in Figure 10.73. This graph, like the one in Figure 10.71, is a limaçon.

CHECKPOINT Now try Exercise 29.

Some curves reach their zeros and maximum r-values at more than one point, as shown in Example 4.

Example 4 Sketching a Polar Graph

Sketch the graph of $r = 2 \cos 3\theta$.

Solution

Symmetry: With respect to the polar axis

Maximum value of $|r|$*:* $|r| = 2$ when $3\theta = 0, \pi, 2\pi, 3\pi$ or $\theta = 0, \pi/3, 2\pi/3, \pi$

Zeros of r: $r = 0$ when $3\theta = \pi/2, 3\pi/2, 5\pi/2$ or $\theta = \pi/6, \pi/2, 5\pi/6$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0

By plotting these points and using the specified symmetry, zeros, and maximum values, you can obtain the graph shown in Figure 10.74. This graph is called a **rose curve,** and each of the loops on the graph is called a *petal* of the rose curve. Note how the entire curve is generated as θ increases from 0 to π.

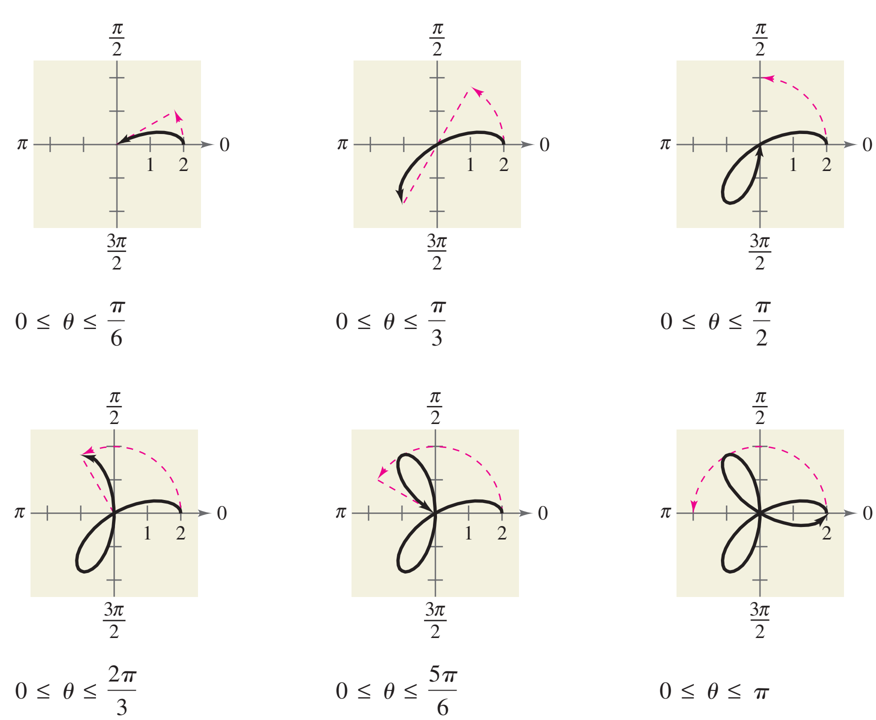


FIGURE 10.74

CHECKPOINT Now try Exercise 33.

Exploration

Notice that the rose curve in Example 4 has three petals. How many petals do the rose curves given by $r = 2 \cos 4\theta$ and $r = 2 \sin 3\theta$ have? Determine the numbers of petals for the curves given by $r = 2 \cos n\theta$ and $r = 2 \sin n\theta$, where n is a positive integer.

Technology

Use a graphing utility in *polar* mode to verify the graph of $r = 2 \cos 3\theta$ shown in Figure 10.74.

Special Polar Graphs

Several important types of graphs have equations that are simpler in polar form than in rectangular form. For example, the circle

$$r = 4 \sin \theta$$

in Example 1 has the more complicated rectangular equation

$$x^2 + (y - 2)^2 = 4.$$

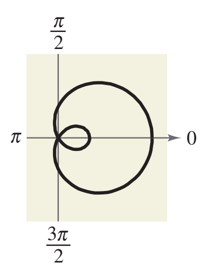
Several other types of graphs that have simple polar equations are shown below.

Limaçons

$r = a \pm b \cos \theta$

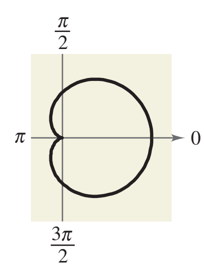
$r = a \pm b \sin \theta$

$(a > 0, b > 0)$



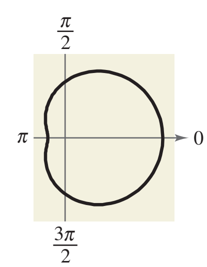
$\frac{a}{b} < 1$

Limaçon with inner loop



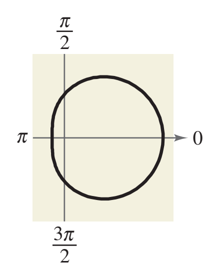
$\frac{a}{b} = 1$

Cardioid (heart-shaped)



$1 < \frac{a}{b} < 2$

Dimpled limaçon

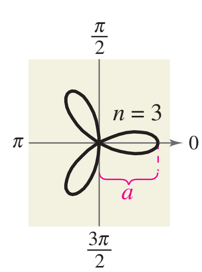


$\frac{a}{b} \geq 2$

Convex limaçon

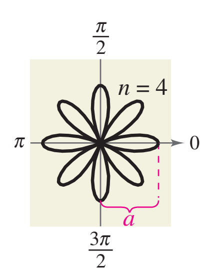
Rose Curves

n petals if n is odd,
$2n$ petals if n is even
$(n \geq 2)$



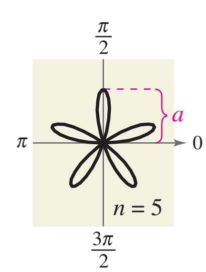
$r = a \cos n\theta$

Rose curve



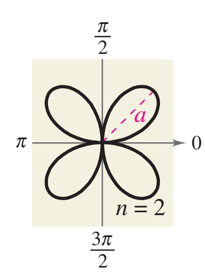
$r = a \cos n\theta$

Rose curve



$r = a \sin n\theta$

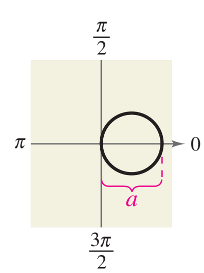
Rose curve



$r = a \sin n\theta$

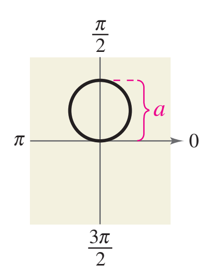
Rose curve

Circles and Lemniscates



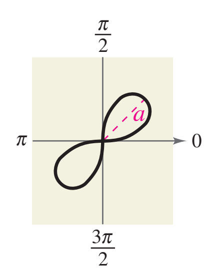
$r = a \cos \theta$

Circle



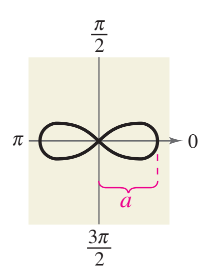
$r = a \sin \theta$

Circle



$r^2 = a^2 \sin 2\theta$

Lemniscate



$r^2 = a^2 \cos 2\theta$

Lemniscate

The quick tests for symmetry presented in this section can be especially useful in graphing these special polar graphs.

Example 5 Sketching a Rose Curve

Sketch the graph of $r = 3 \cos 2\theta$.

Solution

Type of curve: Rose curve with $2n = 4$ petals

Symmetry: With respect to polar axis, the line $\theta = \pi/2$, and the pole

Maximum value of $|r|$: $|r| = 3$ when $\theta = 0, \pi/2, \pi, 3\pi/2$

Zeros of r: $r = 0$ when $\theta = \pi/4, 3\pi/4$

Using this information together with the additional points shown in the following table, you obtain the graph shown in Figure 10.75.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
r	3	$\frac{3}{2}$	0	$-\frac{3}{2}$

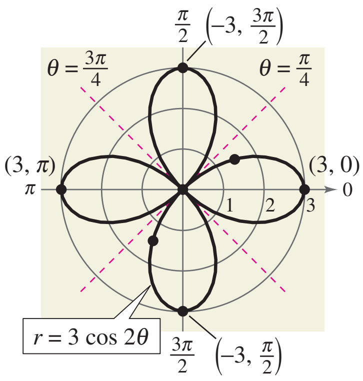


FIGURE 10.75

CHECKPOINT Now try Exercise 35.

Example 6 Sketching a Lemniscate

Sketch the graph of $r^2 = 9 \sin 2\theta$.

Solution

Type of curve: Lemniscate

Symmetry: With respect to the pole

Maximum value of $|r|$: $|r| = 3$ when $\theta = \dfrac{\pi}{4}$

Zeros of r: $r = 0$ when $\theta = 0, \dfrac{\pi}{2}$

If $\sin 2\theta < 0$, this equation has no solution points. So, you restrict the values of θ to those for which $\sin 2\theta \geq 0$.

$$0 \leq \theta \leq \frac{\pi}{2} \qquad \text{or} \qquad \pi \leq \theta \leq \frac{3\pi}{2}$$

Moreover, using symmetry, you need to consider only the first of these two intervals. By finding a few additional points (see table below), you can obtain the graph shown in Figure 10.76.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{4}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$r = \pm 3\sqrt{\sin 2\theta}$	0	$\frac{\pm 3}{\sqrt{2}}$	± 3	$\frac{\pm 3}{\sqrt{2}}$	0

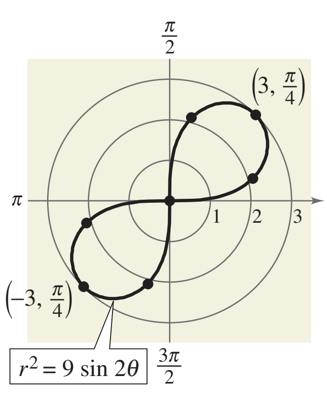


FIGURE 10.76

CHECKPOINT Now try Exercise 39.

Activities

1. Test $r^2 = 3 \sin \theta$ for symmetry.
   Answer: Symmetric with respect to the pole
2. Find the maximum value of $|r|$ and any zeros of r for the polar equation $r = 2 + 2 \cos \theta$.
   Answer: Maximum value of $|r| = 4$ when $\theta = 0$, and $r = 0$ when $\theta = \pi$
3. Identify the shape of the graph of the polar equation $r = 2 \sin 3\theta$.
   Answer: Rose curve with 3 petals

10.8 Exercises

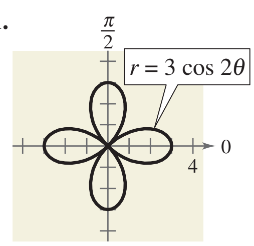
VOCABULARY CHECK: Fill in the blanks.

1. The graph of $r = f(\sin\theta)$ is symmetric with respect to the line ________.
2. The graph of $r = g(\cos\theta)$ is symmetric with respect to the ________ ________.
3. The equation $r = 2 + \cos\theta$ represents a ________ ________.
4. The equation $r = 2\cos\theta$ represents a ________.
5. The equation $r^2 = 4\sin 2\theta$ represents a ________.
6. The equation $r = 1 + \sin\theta$ represents a ________.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–6, identify the type of polar graph.

1.



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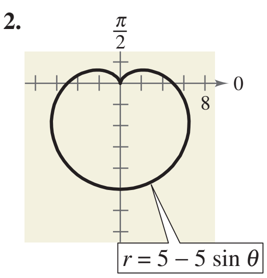

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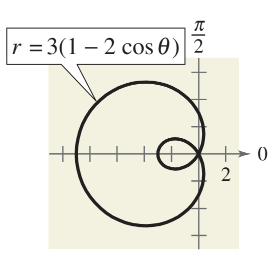

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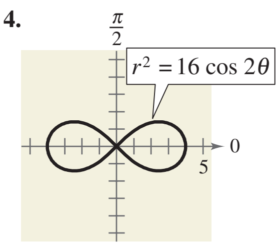
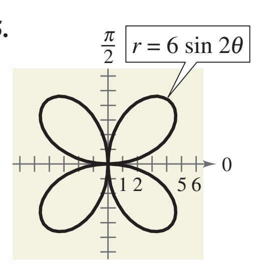
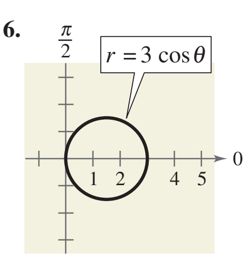


5.



6.



In Exercises 7–12, test for symmetry with respect to $\theta = \pi/2$, the polar axis, and the pole.

7. $r = 5 + 4\cos\theta$

8. $r = 16\cos 3\theta$

9. $r = \dfrac{2}{1 + \sin\theta}$

10. $r = \dfrac{3}{2 + \cos\theta}$

11. $r^2 = 16\cos 2\theta$

12. $r^2 = 36\sin 2\theta$

In Exercises 13–16, find the maximum value of $|r|$ and any zeros of r.

13. $r = 10(1 - \sin\theta)$

14. $r = 6 + 12\cos\theta$

15. $r = 4\cos 3\theta$

16. $r = 3\sin 2\theta$

In Exercises 17–40, sketch the graph of the polar equation using symmetry, zeros, maximum r-values, and any other additional points.

17. $r = 5$

18. $r = 2$

19. $r = \dfrac{\pi}{6}$

20. $r = -\dfrac{3\pi}{4}$

21. $r = 3\sin\theta$

22. $r = 4\cos\theta$

23. $r = 3(1 - \cos\theta)$

24. $r = 4(1 - \sin\theta)$

25. $r = 4(1 + \sin\theta)$

26. $r = 2(1 + \cos\theta)$

27. $r = 3 + 6\sin\theta$

28. $r = 4 - 3\sin\theta$

29. $r = 1 - 2\sin\theta$

30. $r = 1 - 2\cos\theta$

31. $r = 3 - 4\cos\theta$

32. $r = 4 + 3\cos\theta$

33. $r = 5\sin 2\theta$

34. $r = 3\cos 2\theta$

35. $r = 2\sec\theta$

36. $r = 5\csc\theta$

37. $r = \dfrac{3}{\sin\theta - 2\cos\theta}$

38. $r = \dfrac{6}{2\sin\theta - 3\cos\theta}$

39. $r^2 = 9\cos 2\theta$

40. $r^2 = 4\sin\theta$

In Exercises 41–46, use a graphing utility to graph the polar equation. Describe your viewing window.

41. $r = 8\cos\theta$

42. $r = \cos 2\theta$

43. $r = 3(2 - \sin\theta)$

44. $r = 2\cos(3\theta - 2)$

45. $r = 8\sin\theta\cos^2\theta$

46. $r = 2\csc\theta + 5$

In Exercises 47–52, use a graphing utility to graph the polar equation. Find an interval for θ for which the graph is traced *only once*.

47. $r = 3 - 4\cos\theta$

48. $r = 5 + 4\cos\theta$

49. $r = 2\cos\left(\frac{3\theta}{2}\right)$ **50.** $r = 3\sin\left(\frac{5\theta}{2}\right)$

51. $r^2 = 9\sin 2\theta$ **52.** $r^2 = \frac{1}{\theta}$

In Exercises 53–56, use a graphing utility to graph the polar equation and show that the indicated line is an asymptote of the graph.

	Name of Graph	*Polar Equation*	*Asymptote*
53.	Conchoid	$r = 2 - \sec\theta$	$x = -1$
54.	Conchoid	$r = 2 + \csc\theta$	$y = 1$
55.	Hyperbolic spiral	$r = \frac{3}{\theta}$	$y = 3$
56.	Strophoid	$r = 2\cos 2\theta \sec\theta$	$x = -2$

Synthesis

True or False? **In Exercises 57 and 58, determine whether the statement is true or false. Justify your answer.**

57. In the polar coordinate system, if a graph that has symmetry with respect to the polar axis were folded on the line $\theta = 0$, the portion of the graph above the polar axis would coincide with the portion of the graph below the polar axis.

58. In the polar coordinate system, if a graph that has symmetry with respect to the pole were folded on the line $\theta = 3\pi/4$, the portion of the graph on one side of the fold would coincide with the portion of the graph on the other side of the fold.

59. ***Exploration*** Sketch the graph of $r = 6\cos\theta$ over each interval. Describe the part of the graph obtained in each case.

(a) $0 \le \theta \le \frac{\pi}{2}$ (b) $\frac{\pi}{2} \le \theta \le \pi$

(c) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ (d) $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

60. ***Graphical Reasoning*** Use a graphing utility to graph the polar equation $r = 6[1 + \cos(\theta - \phi)]$ for (a) $\phi = 0$, (b) $\phi = \pi/4$, and (c) $\phi = \pi/2$. Use the graphs to describe the effect of the angle ϕ. Write the equation as a function of $\sin\theta$ for part (c).

61. The graph of $r = f(\theta)$ is rotated about the pole through an angle ϕ. Show that the equation of the rotated graph is $r = f(\theta - \phi)$.

62. Consider the graph of $r = f(\sin\theta)$.

(a) Show that if the graph is rotated counterclockwise $\pi/2$ radians about the pole, the equation of the rotated graph is $r = f(-\cos\theta)$.

(b) Show that if the graph is rotated counterclockwise π radians about the pole, the equation of the rotated graph is $r = f(-\sin\theta)$.

(c) Show that if the graph is rotated counterclockwise $3\pi/2$ radians about the pole, the equation of the rotated graph is $r = f(\cos\theta)$.

In Exercises 63–66, use the results of Exercises 61 and 62.

63. Write an equation for the limaçon $r = 2 - \sin\theta$ after it has been rotated through the given angle.

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{2}$

64. Write an equation for the rose curve $r = 2\sin 2\theta$ after it has been rotated through the given angle.

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) π

65. Sketch the graph of each equation.

(a) $r = 1 - \sin\theta$ (b) $r = 1 - \sin\left(\theta - \frac{\pi}{4}\right)$

66. Sketch the graph of each equation.

(a) $r = 3\sec\theta$ (b) $r = 3\sec\left(\theta - \frac{\pi}{4}\right)$

(c) $r = 3\sec\left(\theta + \frac{\pi}{3}\right)$ (d) $r = 3\sec\left(\theta - \frac{\pi}{2}\right)$

67. ***Exploration*** Use a graphing utility to graph and identify $r = 2 + k\sin\theta$ for $k = 0, 1, 2$, and 3.

68. ***Exploration*** Consider the equation $r = 3\sin k\theta$.

(a) Use a graphing utility to graph the equation for $k = 1.5$. Find the interval for θ over which the graph is traced only once.

(b) Use a graphing utility to graph the equation for $k = 2.5$. Find the interval for θ over which the graph is traced only once.

(c) Is it possible to find an interval for θ over which the graph is traced only once for any rational number k? Explain.

Skills Review

In Exercises 69–72, find the zeros (if any) of the rational function.

69. $f(x) = \frac{x^2 - 9}{x + 1}$ **70.** $f(x) = 6 + \frac{4}{x^2 + 4}$

71. $f(x) = 5 - \frac{3}{x - 2}$ **72.** $f(x) = \frac{x^3 - 27}{x^2 + 4}$

In Exercises 73 and 74, find the standard form of the equation of the ellipse with the given characteristics. Then sketch the ellipse.

73. Vertices: $(-4, 2), (2, 2)$; minor axis of length 4

74. Foci: $(3, 2), (3, -4)$; major axis of length 8