10.5 Rotation of Conics

What you should learn

- Rotate the coordinate axes to eliminate the *xy*-term in equations of conics.
- Use the discriminant to classify conics.

Why you should learn it

As illustrated in Exercises 7–18 on page 769, rotation of the coordinate axes can help you identify the graph of a general second-degree equation.

Rotation

In the preceding section, you learned that the equation of a conic with axes parallel to one of the coordinate axes has a standard form that can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \qquad \text{Horizontal or vertical axis}$$

In this section, you will study the equations of conics whose axes are rotated so that they are not parallel to either the x-axis or the y-axis. The general equation for such conics contains an xy-term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad \text{Equation in } xy\text{-plane}$$

To eliminate this xy-term, you can use a procedure called **rotation of axes.** The objective is to rotate the x- and y-axes until they are parallel to the axes of the conic. The rotated axes are denoted as the x'-axis and the y'-axis, as shown in Figure 10.42.

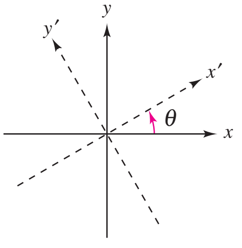


FIGURE 10.42

After the rotation, the equation of the conic in the new $x'y'$-plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0. \qquad \text{Equation in } x'y'\text{-plane}$$

Because this equation has no xy-term, you can obtain a standard form by completing the square. The following theorem identifies how much to rotate the axes to eliminate the xy-term and also the equations for determining the new coefficients A', C', D', E', and F'.

Rotation of Axes to Eliminate an *xy*-Term

The general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

The coefficients of the new equation are obtained by making the substitutions $x = x' \cos \theta - y' \sin \theta$ and $y = x' \sin \theta + y' \cos \theta$.

STUDY TIP

Remember that the substitutions

$$x = x' \cos\theta - y' \sin\theta$$

and

$$y = x' \sin\theta + y' \cos\theta$$

were developed to eliminate the $x'y'$-term in the rotated system. You can use this as a check on your work. In other words, if your final equation contains an $x'y'$-term, you know that you made a mistake.

Example 1 Rotation of Axes for a Hyperbola

Write the equation $xy - 1 = 0$ in standard form.

Solution

Because $A = 0$, $B = 1$, and $C = 0$, you have

$$\cot 2\theta = \frac{A - C}{B} = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

which implies that

$$\begin{aligned} x &= x' \cos\frac{\pi}{4} - y' \sin\frac{\pi}{4} \\ &= x'\left(\frac{1}{\sqrt{2}}\right) - y'\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{x' - y'}{\sqrt{2}} \end{aligned}$$

and

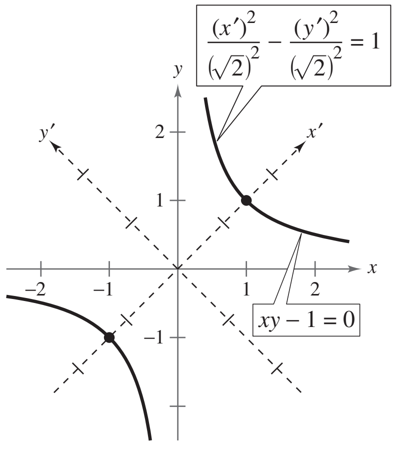
$$\begin{aligned} y &= x' \sin\frac{\pi}{4} + y' \cos\frac{\pi}{4} \\ &= x'\left(\frac{1}{\sqrt{2}}\right) + y'\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{x' + y'}{\sqrt{2}}. \end{aligned}$$

The equation in the $x'y'$-system is obtained by substituting these expressions in the equation $xy - 1 = 0$.

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 1 = 0$$

$$\frac{(x')^2 - (y')^2}{2} - 1 = 0$$

$$\frac{(x')^2}{(\sqrt{2})^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1 \qquad \text{Write in standard form.}$$



Vertices:
In $x'y'$-system: $(\sqrt{2}, 0), (-\sqrt{2}, 0)$
In xy-system: $(1, 1), (-1, -1)$
FIGURE 10.43

In the $x'y'$-system, this is a hyperbola centered at the origin with vertices at $(\pm\sqrt{2}, 0)$, as shown in Figure 10.43. To find the coordinates of the vertices in the xy-system, substitute the coordinates $(\pm\sqrt{2}, 0)$ in the equations

$$x = \frac{x' - y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x' + y'}{\sqrt{2}}.$$

This substitution yields the vertices $(1, 1)$ and $(-1, -1)$ in the xy-system. Note also that the asymptotes of the hyperbola have equations $y' = \pm x'$, which correspond to the original x- and y-axes.

CHECKPOINT Now try Exercise 7.

Example 2 Rotation of Axes for an Ellipse

Sketch the graph of $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$.

Solution

Because $A = 7$, $B = -6\sqrt{3}$, and $C = 13$, you have

$$\cot 2\theta = \frac{A - C}{B} = \frac{7 - 13}{-6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

which implies that $\theta = \pi/6$. The equation in the $x'y'$-system is obtained by making the substitutions

$$\begin{aligned} x &= x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} \\ &= x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{3}x' - y'}{2} \end{aligned}$$

and

$$\begin{aligned} y &= x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6} \\ &= x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{x' + \sqrt{3}y'}{2} \end{aligned}$$

in the original equation. So, you have

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$

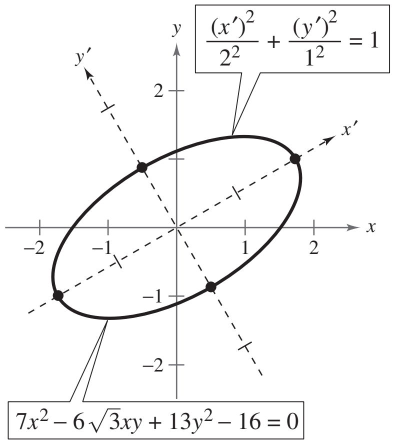
$$7\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + 13\left(\frac{x' + \sqrt{3}y'}{2}\right)^2 - 16 = 0$$

which simplifies to

$$\begin{aligned} 4(x')^2 + 16(y')^2 - 16 &= 0 \\ 4(x')^2 + 16(y')^2 &= 16 \\ \frac{(x')^2}{4} + \frac{(y')^2}{1} &= 1 \\ \frac{(x')^2}{2^2} + \frac{(y')^2}{1^2} &= 1. \end{aligned}$$ Write in standard form.

This is the equation of an ellipse centered at the origin with vertices $(\pm 2, 0)$ in the $x'y'$-system, as shown in Figure 10.44.

CHECKPOINT Now try Exercise 13.



Vertices:
In $x'y'$-system: $(\pm 2, 0)$, $(0, \pm 1)$
In xy-system: $(\sqrt{3}, 1)$, $(-\sqrt{3}, -1)$, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

FIGURE 10.44

Example 3 Rotation of Axes for a Parabola

Sketch the graph of $x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0$.

Solution

Because $A = 1$, $B = -4$, and $C = 4$, you have

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 4}{-4} = \frac{3}{4}.$$

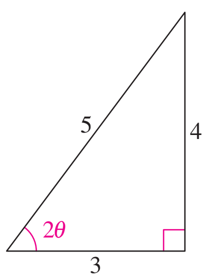


FIGURE 10.45

Using this information, draw a right triangle as shown in Figure 10.45. From the figure, you can see that $\cos 2\theta = \frac{3}{5}$. To find the values of $\sin \theta$ and $\cos \theta$, you can use the half-angle formulas in the forms

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} \quad \text{and} \quad \cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}}.$$

So,

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

$$\cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}.$$

Consequently, you use the substitutions

$$x = x' \cos \theta - y' \sin \theta$$
$$= x'\left(\frac{2}{\sqrt{5}}\right) - y'\left(\frac{1}{\sqrt{5}}\right) = \frac{2x' - y'}{\sqrt{5}}$$

$$y = x' \sin \theta + y' \cos \theta$$
$$= x'\left(\frac{1}{\sqrt{5}}\right) + y'\left(\frac{2}{\sqrt{5}}\right) = \frac{x' + 2y'}{\sqrt{5}}.$$

Substituting these expressions in the original equation, you have

$$x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0$$

$$\left(\frac{2x' - y'}{\sqrt{5}}\right)^2 - 4\left(\frac{2x' - y'}{\sqrt{5}}\right)\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 4\left(\frac{x' + 2y'}{\sqrt{5}}\right)^2 + 5\sqrt{5}\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 1 = 0$$

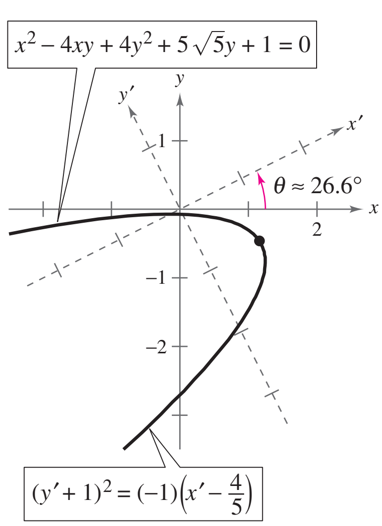
which simplifies as follows.

$$5(y')^2 + 5x' + 10y' + 1 = 0$$

$$5[(y')^2 + 2y'] = -5x' - 1 \qquad \text{Group terms.}$$

$$5(y' + 1)^2 = -5x' + 4 \qquad \text{Write in completed square form.}$$

$$(y' + 1)^2 = (-1)\left(x' - \frac{4}{5}\right) \qquad \text{Write in standard form.}$$



Vertex:
In $x'y'$-system: $\left(\frac{4}{5}, -1\right)$
In xy-system: $\left(\frac{13}{5\sqrt{5}}, -\frac{6}{5\sqrt{5}}\right)$

FIGURE 10.46

The graph of this equation is a parabola with vertex $\left(\frac{4}{5}, -1\right)$. Its axis is parallel to the x'-axis in the $x'y'$-system, and because $\sin \theta = 1/\sqrt{5}$, $\theta \approx 26.6°$, as shown in Figure 10.46.

CHECKPOINT Now try Exercise 17.

Invariants Under Rotation

In the rotation of axes theorem listed at the beginning of this section, note that the constant term is the same in both equations, $F' = F$. Such quantities are **invariant under rotation.** The next theorem lists some other rotation invariants.

Rotation Invariants

The rotation of the coordinate axes through an angle θ that transforms the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

has the following rotation invariants.

1. $F = F'$

2. $A + C = A' + C'$

3. $B^2 - 4AC = (B')^2 - 4A'C'$

STUDY TIP

If there is an xy-term in the equation of a conic, you should realize then that the conic is rotated. Before rotating the axes, you should use the discriminant to classify the conic.

You can use the results of this theorem to classify the graph of a second-degree equation *with* an xy-term in much the same way you do for a second-degree equation *without* an xy-term. Note that because $B' = 0$, the invariant $B^2 - 4AC$ reduces to

$$B^2 - 4AC = -4A'C'. \qquad \text{Discriminant}$$

This quantity is called the **discriminant** of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Now, from the classification procedure given in Section 10.4, you know that the sign of $A'C'$ determines the type of graph for the equation

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0.$$

Consequently, the sign of $B^2 - 4AC$ will determine the type of graph for the original equation, as given in the following classification.

Classification of Conics by the Discriminant

The graph of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is, except in degenerate cases, determined by its discriminant as follows.

1. *Ellipse or circle:* $B^2 - 4AC < 0$

2. *Parabola:* $B^2 - 4AC = 0$

3. *Hyperbola:* $B^2 - 4AC > 0$

For example, in the general equation

$$3x^2 + 7xy + 5y^2 - 6x - 7y + 15 = 0$$

you have $A = 3$, $B = 7$, and $C = 5$. So the discriminant is

$$B^2 - 4AC = 7^2 - 4(3)(5) = 49 - 60 = -11.$$

Because $-11 < 0$, the graph of the equation is an ellipse or a circle.

Activities

1. Find the angle of rotation necessary to eliminate the xy-term in the equation $3x^2 + 2xy - y^2 + x - 1 = 0$.
   Answer: $\theta \approx 13.28°$
2. Rotate the axes to eliminate the xy-term in the equation $x^2 + 2xy + y^2 - \sqrt{2}y = 0$.
   Answer: $2(x')^2 - x' - y' = 0,\ \theta = \frac{\pi}{4}$
3. Use the discriminant to classify each equation.
   a. $x^2 - 4xy + y^2 + 2x - 3y + 1 = 0$
   b. $4x^2 - xy + 4y^2 + 2x - 3y + 1 = 0$
   c. $x^2 - xy + 4y^2 + 2x - 3y + 1 = 0$
   d. $x^2 - 4xy + 4y^2 + 2x - 3y + 1 = 0$
   Answer: (a) Hyperbola, (b) Ellipse or circle, (c) Ellipse or circle, (d) Parabola

Example 4 Rotation and Graphing Utilities

For each equation, classify the graph of the equation, use the Quadratic Formula to solve for y, and then use a graphing utility to graph the equation.

a. $2x^2 - 3xy + 2y^2 - 2x = 0$ **b.** $x^2 - 6xy + 9y^2 - 2y + 1 = 0$

c. $3x^2 + 8xy + 4y^2 - 7 = 0$

Solution

a. Because $B^2 - 4AC = 9 - 16 < 0$, the graph is a circle or an ellipse. Solve for y as follows.

$$2x^2 - 3xy + 2y^2 - 2x = 0 \qquad \text{Write original equation.}$$

$$2y^2 - 3xy + (2x^2 - 2x) = 0 \qquad \text{Quadratic form } ay^2 + by + c = 0$$

$$y = \frac{-(-3x) \pm \sqrt{(-3x)^2 - 4(2)(2x^2 - 2x)}}{2(2)}$$

$$y = \frac{3x \pm \sqrt{x(16 - 7x)}}{4}$$

Graph both of the equations to obtain the ellipse shown in Figure 10.47.

$$y_1 = \frac{3x + \sqrt{x(16 - 7x)}}{4} \qquad \text{Top half of ellipse}$$

$$y_2 = \frac{3x - \sqrt{x(16 - 7x)}}{4} \qquad \text{Bottom half of ellipse}$$

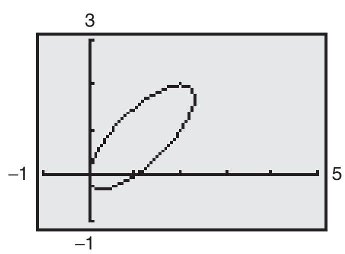


FIGURE 10.47

b. Because $B^2 - 4AC = 36 - 36 = 0$, the graph is a parabola.

$$x^2 - 6xy + 9y^2 - 2y + 1 = 0 \qquad \text{Write original equation.}$$

$$9y^2 - (6x + 2)y + (x^2 + 1) = 0 \qquad \text{Quadratic form } ay^2 + by + c = 0$$

$$y = \frac{(6x + 2) \pm \sqrt{(6x + 2)^2 - 4(9)(x^2 + 1)}}{2(9)}$$

Graphing both of the equations to obtain the parabola shown in Figure 10.48.

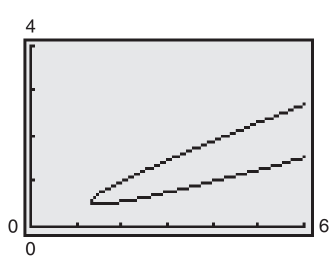


FIGURE 10.48

c. Because $B^2 - 4AC = 64 - 48 > 0$, the graph is a hyperbola.

$$3x^2 + 8xy + 4y^2 - 7 = 0 \qquad \text{Write original equation.}$$

$$4y^2 + 8xy + (3x^2 - 7) = 0 \qquad \text{Quadratic form } ay^2 + by + c = 0$$

$$y = \frac{-8x \pm \sqrt{(8x)^2 - 4(4)(3x^2 - 7)}}{2(4)}$$

The graphs of these two equations yield the hyperbola shown in Figure 10.49.

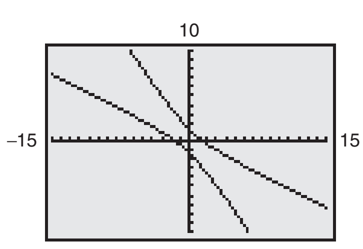


FIGURE 10.49

CHECKPOINT Now try Exercise 33.

Writing about Mathematics

Classifying a Graph as a Hyperbola In Section 2.6, it was mentioned that the graph of $f(x) = 1/x$ is a hyperbola. Use the techniques in this section to verify this, and justify each step. Compare your results with those of another student.

10.5 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The procedure used to eliminate the xy-term in a general second-degree equation is called ________ of ________.
2. After rotating the coordinate axes through an angle θ, the general second-degree equation in the new $x'y'$-plane will have the form ________.
3. Quantities that are equal in both the original equation of a conic and the equation of the rotated conic are ________ ________ ________.
4. The quantity $B^2 - 4AC$ is called the ________ of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–6, the $x'y'$-coordinate system has been rotated θ degrees from the xy-coordinate system. The coordinates of a point in the xy-coordinate system are given. Find the coordinates of the point in the rotated coordinate system.

1. $\theta = 90°$, $(0, 3)$
2. $\theta = 45°$, $(3, 3)$
3. $\theta = 30°$, $(1, 3)$
4. $\theta = 60°$, $(3, 1)$
5. $\theta = 45°$, $(2, 1)$
6. $\theta = 30°$, $(2, 4)$

In Exercises 7–18, rotate the axes to eliminate the xy-term in the equation. Then write the equation in standard form. Sketch the graph of the resulting equation, showing both sets of axes.

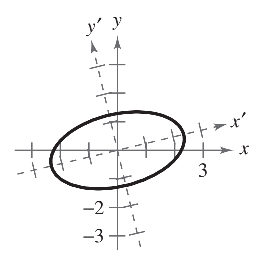
7. $xy + 1 = 0$
8. $xy - 2 = 0$
9. $x^2 - 2xy + y^2 - 1 = 0$
10. $xy + x - 2y + 3 = 0$
11. $xy - 2y - 4x = 0$
12. $2x^2 - 3xy - 2y^2 + 10 = 0$
13. $5x^2 - 6xy + 5y^2 - 12 = 0$
14. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$
15. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$
16. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$
17. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$
18. $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$

In Exercises 19–26, use a graphing utility to graph the conic. Determine the angle θ through which the axes are rotated. Explain how you used the graphing utility to obtain the graph.

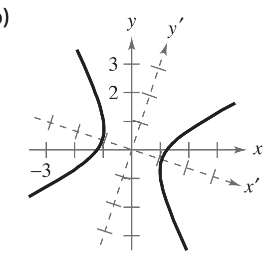
19. $x^2 + 2xy + y^2 = 20$
20. $x^2 - 4xy + 2y^2 = 6$
21. $17x^2 + 32xy - 7y^2 = 75$
22. $40x^2 + 36xy + 25y^2 = 52$
23. $32x^2 + 48xy + 8y^2 = 50$
24. $24x^2 + 18xy + 12y^2 = 34$
25. $4x^2 - 12xy + 9y^2 + (4\sqrt{13} - 12)x - (6\sqrt{13} + 8)y = 91$
26. $6x^2 - 4xy + 8y^2 + (5\sqrt{5} - 10)x - (7\sqrt{5} + 5)y = 80$

In Exercises 27–32, match the graph with its equation. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)



(b)



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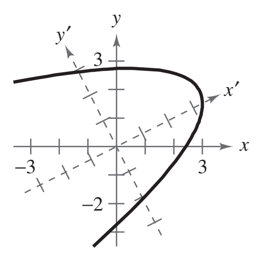

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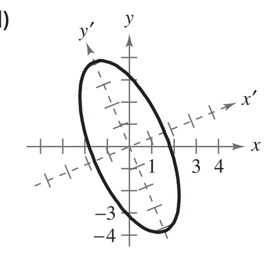

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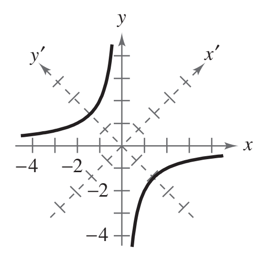
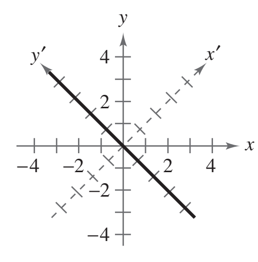


(f)



27. $xy + 2 = 0$

28. $x^2 + 2xy + y^2 = 0$

29. $-2x^2 + 3xy + 2y^2 + 3 = 0$

30. $x^2 - xy + 3y^2 - 5 = 0$

31. $3x^2 + 2xy + y^2 - 10 = 0$

32. $x^2 - 4xy + 4y^2 + 10x - 30 = 0$

In Exercises 33–40, (a) use the discriminant to classify the graph, (b) use the Quadratic Formula to solve for *y*, and (c) use a graphing utility to graph the equation.

33. $16x^2 - 8xy + y^2 - 10x + 5y = 0$

34. $x^2 - 4xy - 2y^2 - 6 = 0$

35. $12x^2 - 6xy + 7y^2 - 45 = 0$

36. $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$

37. $x^2 - 6xy - 5y^2 + 4x - 22 = 0$

38. $36x^2 - 60xy + 25y^2 + 9y = 0$

39. $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$

40. $x^2 + xy + 4y^2 + x + y - 4 = 0$

In Exercises 41–44, sketch (if possible) the graph of the degenerate conic.

41. $y^2 - 9x^2 = 0$

42. $x^2 + y^2 - 2x + 6y + 10 = 0$

43. $x^2 + 2xy + y^2 - 1 = 0$

44. $x^2 - 10xy + y^2 = 0$

In Exercises 45–58, find any points of intersection of the graphs algebraically and then verify using a graphing utility.

45. $-x^2 + y^2 + 4x - 6y + 4 = 0$
$x^2 + y^2 - 4x - 6y + 12 = 0$

46. $-x^2 - y^2 - 8x + 20y - 7 = 0$
$x^2 + 9y^2 + 8x + 4y + 7 = 0$

47. $-4x^2 - y^2 - 16x + 24y - 16 = 0$
$4x^2 + y^2 + 40x - 24y + 208 = 0$

48. $x^2 - 4y^2 - 20x - 64y - 172 = 0$
$16x^2 + 4y^2 - 320x + 64y + 1600 = 0$

49. $x^2 - y^2 - 12x + 16y - 64 = 0$
$x^2 + y^2 - 12x - 16y + 64 = 0$

50. $x^2 + 4y^2 - 2x - 8y + 1 = 0$
$-x^2 + 2x - 4y - 1 = 0$

51. $-16x^2 - y^2 + 24y - 80 = 0$
$16x^2 + 25y^2 - 400 = 0$

52. $16x^2 - y^2 + 16y - 128 = 0$
$y^2 - 48x - 16y - 32 = 0$

53. $x^2 + y^2 - 4 = 0$
$3x - y^2 = 0$

54. $4x^2 + 9y^2 - 36y = 0$
$x^2 + 9y - 27 = 0$

55. $x^2 + 2y^2 - 4x + 6y - 5 = 0$
$-x + y - 4 = 0$

56. $x^2 + 2y^2 - 4x + 6y - 5 = 0$
$x^2 - 4x - y + 4 = 0$

57. $xy + x - 2y + 3 = 0$
$x^2 + 4y^2 - 9 = 0$

58. $5x^2 - 2xy + 5y^2 - 12 = 0$
$x + y - 1 = 0$

Synthesis

True or False? **In Exercises 59 and 60, determine whether the statement is true or false. Justify your answer.**

59. The graph of the equation

$$x^2 + xy + ky^2 + 6x + 10 = 0$$

where k is any constant less than $\frac{1}{4}$, is a hyperbola.

60. After a rotation of axes is used to eliminate the xy-term from an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the coefficients of the x^2- and y^2-terms remain A and C, respectively.

61. Show that the equation

$$x^2 + y^2 = r^2$$

is invariant under rotation of axes.

62. Find the lengths of the major and minor axes of the ellipse graphed in Exercise 14.

Skills Review

In Exercises 63–70, graph the function.

63. $f(x) = |x + 3|$

64. $f(x) = |x - 4| + 1$

65. $g(x) = \sqrt{4 - x^2}$

66. $g(x) = \sqrt{3x - 2}$

67. $h(t) = -(t - 2)^3 + 3$

68. $h(t) = \frac{1}{2}(t + 4)^3$

69. $f(t) = [\![t - 5]\!] + 1$

70. $f(t) = -2[\![t]\!] + 3$

In Exercises 71–74, find the area of the triangle.

71. $C = 110^\circ, a = 8, b = 12$

72. $B = 70^\circ, a = 25, c = 16$

73. $a = 11, b = 18, c = 10$

74. $a = 23, b = 35, c = 27$