10.1 Lines

What you should learn

- Find the inclination of a line.
- Find the angle between two lines.
- Find the distance between a point and a line.

Why you should learn it

The inclination of a line can be used to measure heights indirectly. For instance, in Exercise 56 on page 734, the inclination of a line can be used to determine the change in elevation from the base to the top of the Johnstown Inclined Plane.

AP/Wide World Photos

The *HM mathSpace®* CD-ROM and *Eduspace®* for this text contain additional resources related to the concepts discussed in this chapter.

Inclination of a Line

In Section 1.3, you learned that the graph of the linear equation

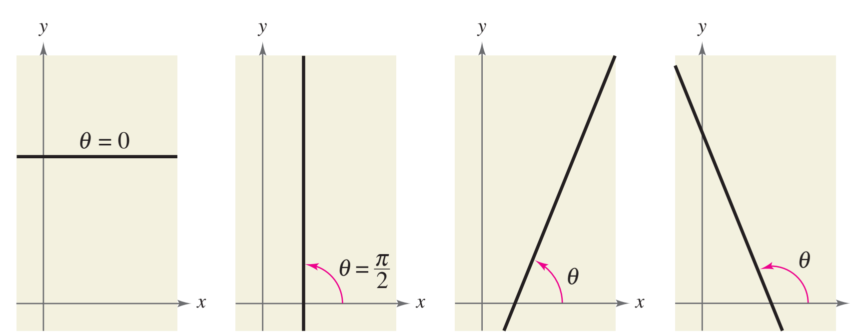
$$y = mx + b$$

is a nonvertical line with slope m and y-intercept $(0, b)$. There, the slope of a line was described as the rate of change in y with respect to x. In this section, you will look at the slope of a line in terms of the angle of inclination of the line.

Every nonhorizontal line must intersect the x-axis. The angle formed by such an intersection determines the **inclination** of the line, as specified in the following definition.

Definition of Inclination

The **inclination** of a nonhorizontal line is the positive angle θ (less than π) measured counterclockwise from the x-axis to the line. (See Figure 10.1.)



Horizontal Line *Vertical Line* *Acute Angle* *Obtuse Angle*

FIGURE 10.1

The inclination of a line is related to its slope in the following manner.

Inclination and Slope

If a nonvertical line has inclination θ and slope m, then

$$m = \tan \theta.$$

For a proof of the relation between inclination and slope, see Proofs in Mathematics on page 806.

Example 1 Finding the Inclination of a Line

Find the inclination of the line $2x + 3y = 6$.

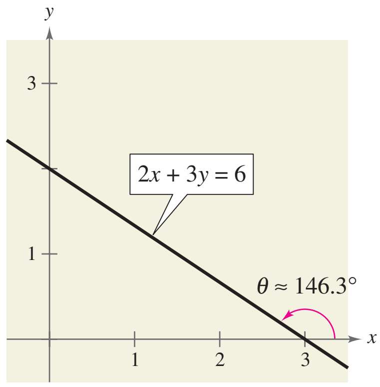


FIGURE 10.2

Solution

The slope of this line is $m = -\frac{2}{3}$. So, its inclination is determined from the equation

$$\tan \theta = -\frac{2}{3}.$$

From Figure 10.2, it follows that $\frac{\pi}{2} < \theta < \pi$. This means that

$$\begin{aligned}\theta &= \pi + \arctan\left(-\frac{2}{3}\right)\\ &\approx \pi + (-0.588)\\ &= \pi - 0.588\\ &\approx 2.554.\end{aligned}$$

The angle of inclination is about 2.554 radians or about 146.3°.

CHECKPOINT Now try Exercise 19.

The Angle Between Two Lines

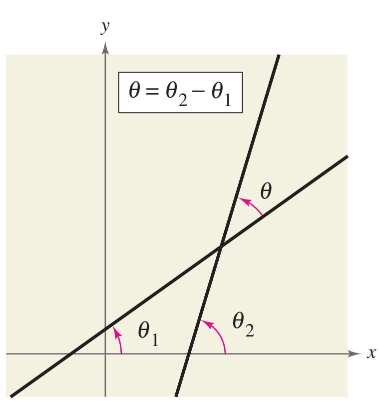


FIGURE 10.3

Two distinct lines in a plane are either parallel or intersecting. If they intersect and are nonperpendicular, their intersection forms two pairs of opposite angles. One pair is acute and the other pair is obtuse. The smaller of these angles is called the **angle between the two lines.** As shown in Figure 10.3, you can use the inclinations of the two lines to find the angle between the two lines. If two lines have inclinations θ_1 and θ_2, where $\theta_1 < \theta_2$ and $\theta_2 - \theta_1 < \pi/2$, the angle between the two lines is

$$\theta = \theta_2 - \theta_1.$$

You can use the formula for the tangent of the difference of two angles

$$\begin{aligned}\tan \theta &= \tan(\theta_2 - \theta_1)\\ &= \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_1 \tan \theta_2}\end{aligned}$$

to obtain the formula for the angle between two lines.

Angle Between Two Lines

If two nonperpendicular lines have slopes m_1 and m_2, the angle between the two lines is

$$\tan \theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|.$$

Example 2 Finding the Angle Between Two Lines

Find the angle between the two lines.

$$\textit{Line 1: } 2x - y - 4 = 0 \qquad \textit{Line 2: } 3x + 4y - 12 = 0$$

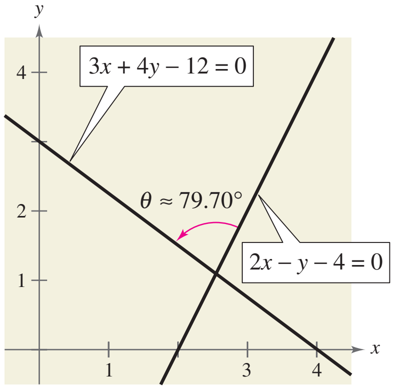


FIGURE 10.4

Solution

The two lines have slopes of $m_1 = 2$ and $m_2 = -\frac{3}{4}$, respectively. So, the tangent of the angle between the two lines is

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{(-3/4) - 2}{1 + (2)(-3/4)}\right| = \left|\frac{-11/4}{-2/4}\right| = \frac{11}{2}.$$

Finally, you can conclude that the angle is

$$\theta = \arctan\frac{11}{2} \approx 1.391 \text{ radians} \approx 79.70°$$

as shown in Figure 10.4.

CHECKPOINT Now try Exercise 27.

The Distance Between a Point and a Line

Finding the distance between a line and a point not on the line is an application of perpendicular lines. This distance is defined as the length of the perpendicular line segment joining the point and the line, as shown in Figure 10.5.

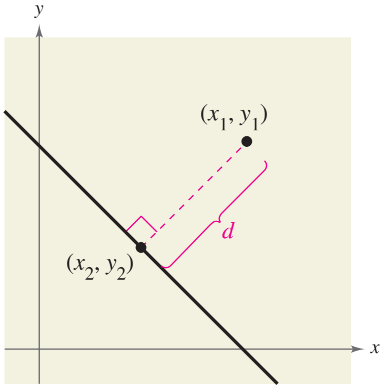


FIGURE 10.5

Distance Between a Point and a Line

The distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Remember that the values of A, B, and C in this distance formula correspond to the general equation of a line, $Ax + By + C = 0$. For a proof of the distance between a point and a line, see Proofs in Mathematics on page 806.

Example 3 Finding the Distance Between a Point and a Line

Find the distance between the point $(4, 1)$ and the line $y = 2x + 1$.

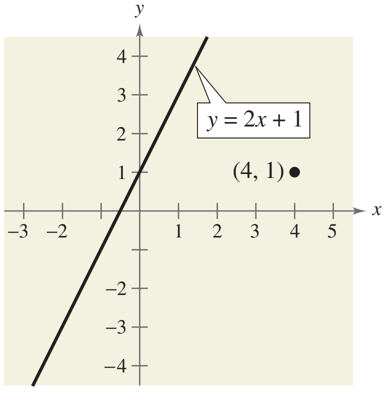


FIGURE 10.6

Solution

The general form of the equation is

$$-2x + y - 1 = 0.$$

So, the distance between the point and the line is

$$d = \frac{|-2(4) + 1(1) + (-1)|}{\sqrt{(-2)^2 + 1^2}} = \frac{8}{\sqrt{5}} \approx 3.58 \text{ units.}$$

The line and the point are shown in Figure 10.6.

CHECKPOINT Now try Exercise 39.

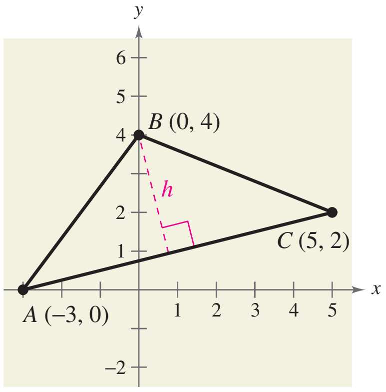


FIGURE 10.7

Example 4 An Application of Two Distance Formulas

Figure 10.7 shows a triangle with vertices $A(-3, 0)$, $B(0, 4)$, and $C(5, 2)$.

a. Find the altitude h from vertex B to side AC.

b. Find the area of the triangle.

Solution

a. To find the altitude, use the formula for the distance between line AC and the point $(0, 4)$. The equation of line AC is obtained as follows.

$$\textit{Slope: } m = \frac{2-0}{5-(-3)} = \frac{2}{8} = \frac{1}{4}$$

$$\textit{Equation: } y - 0 = \frac{1}{4}(x+3) \qquad \text{Point-slope form}$$

$$4y = x + 3 \qquad \text{Multiply each side by 4.}$$

$$x - 4y + 3 = 0 \qquad \text{General form}$$

So, the distance between this line and the point $(0, 4)$ is

$$\text{Altitude} = h = \frac{|1(0) + (-4)(4) + 3|}{\sqrt{1^2 + (-4)^2}} = \frac{13}{\sqrt{17}} \text{ units.}$$

b. Using the formula for the distance between two points, you can find the length of the base AC to be

$$b = \sqrt{[5-(-3)]^2 + (2-0)^2} \qquad \text{Distance Formula}$$

$$= \sqrt{8^2 + 2^2} \qquad \text{Simplify.}$$

$$= \sqrt{68} \qquad \text{Simplify.}$$

$$= 2\sqrt{17} \text{ units.} \qquad \text{Simplify.}$$

Finally, the area of the triangle in Figure 10.7 is

$$A = \frac{1}{2}bh \qquad \text{Formula for the area of a triangle}$$

$$= \frac{1}{2}(2\sqrt{17})\left(\frac{13}{\sqrt{17}}\right) \qquad \text{Substitute for } b \text{ and } h.$$

$$= 13 \text{ square units.} \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 45.

Activities

1. Find the inclination of the line $5x - 4y = 20$.
   Answer: $\theta = 0.896$ radian or $51.34°$
2. Find the angle θ between the lines $x + 2y = 5$ and $3x - y = 6$.
   Answer: $\theta = 1.429$ radians or $81.87°$
3. Find the distance between the point $(3, -1)$ and the line $4x - 3y - 12 = 0$.
   Answer: $d = \frac{3}{5}$ unit

Group Activity: Graphing Utility

Put your students in groups of two. Ask each group to write a graphing calculator program that finds the angle between two lines based on user input. Ask each group to demonstrate the program on an example or exercise in this section.

WRITING ABOUT MATHEMATICS

Inclination and the Angle Between Two Lines Discuss why the inclination of a line can be an angle that is larger than $\pi/2$, but the angle between two lines cannot be larger than $\pi/2$. Decide whether the following statement is true or false: "The inclination of a line is the angle between the line and the x-axis." Explain.

10.1 Exercises

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

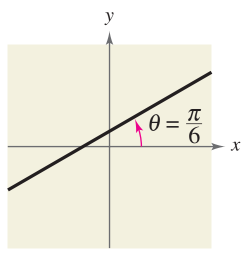
VOCABULARY CHECK: Fill in the blanks.

1. The ________ of a nonhorizontal line is the positive angle θ (less than π) measured counterclockwise from the x-axis to the line.
2. If a nonvertical line has inclination θ and slope m, then $m =$ ________ .
3. If two nonperpendicular lines have slopes m_1 and m_2, the angle between the two lines is $\tan\theta =$ ________ .
4. The distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ is given by $d =$ ________ .

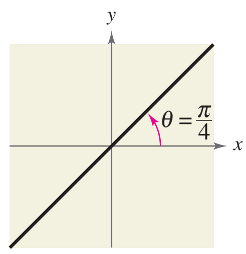
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–8, find the slope of the line with inclination θ.

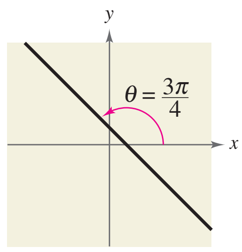
1.



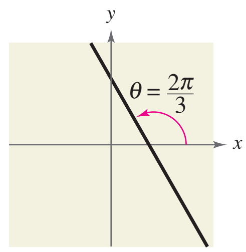
2.



3.



4.



5. $\theta = \dfrac{\pi}{3}$ radians

6. $\theta = \dfrac{5\pi}{6}$ radians

7. $\theta = 1.27$ radians

8. $\theta = 2.88$ radians

In Exercises 9–14, find the inclination θ (in radians and degrees) of the line with a slope of m.

9. $m = -1$

10. $m = -2$

11. $m = 1$

12. $m = 2$

13. $m = \frac{3}{4}$

14. $m = -\frac{5}{2}$

In Exercises 15–18, find the inclination θ (in radians and degrees) of the line passing through the points.

15. $(6, 1), (10, 8)$

16. $(12, 8), (-4, -3)$

17. $(-2, 20), (10, 0)$

18. $(0, 100), (50, 0)$

In Exercises 19–22, find the inclination θ (in radians and degrees) of the line.

19. $6x - 2y + 8 = 0$

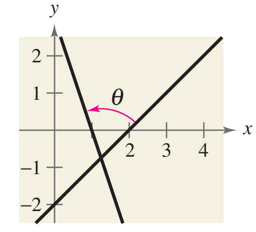
20. $4x + 5y - 9 = 0$

21. $5x + 3y = 0$

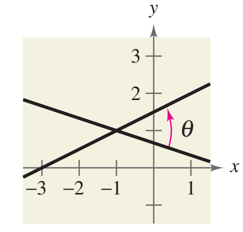
22. $x - y - 10 = 0$

In Exercises 23–32, find the angle θ (in radians and degrees) between the lines.

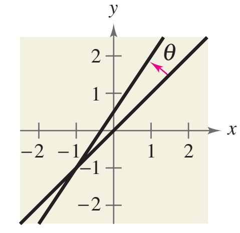
23. $3x + y = 3$
$x - y = 2$



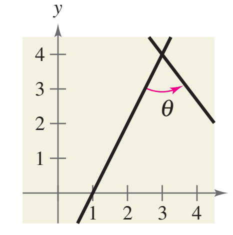
24. $x + 3y = 2$
$x - 2y = -3$



25. $x - y = 0$
$3x - 2y = -1$



26. $2x - y = 2$
$4x + 3y = 24$



27. $x - 2y = 7$
$6x + 2y = 5$

28. $5x + 2y = 16$
$3x - 5y = -1$

29. $x + 2y = 8$
$x - 2y = 2$

30. $3x - 5y = 3$
$3x + 5y = 12$

31. $0.05x - 0.03y = 0.21$

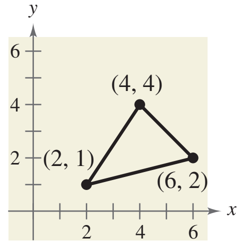
$0.07x + 0.02y = 0.16$

32. $0.02x - 0.05y = -0.19$

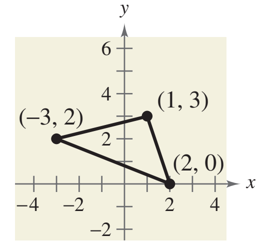
$0.03x + 0.04y = 0.52$

Angle Measurement **In Exercises 33–36, find the slope of each side of the triangle and use the slopes to find the measures of the interior angles.**

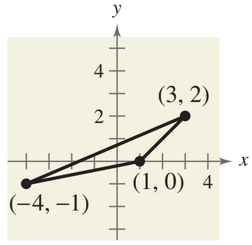
33.



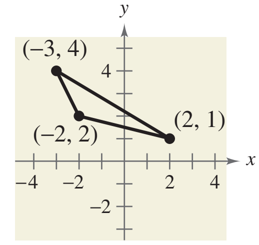
34.



35.



36.



In Exercises 37–44, find the distance between the point and the line.

	Point	*Line*
37.	$(0, 0)$	$4x + 3y = 0$
38.	$(0, 0)$	$2x - y = 4$
39.	$(2, 3)$	$4x + 3y = 10$
40.	$(-2, 1)$	$x - y = 2$
41.	$(6, 2)$	$x + 1 = 0$
42.	$(10, 8)$	$y - 4 = 0$
43.	$(0, 8)$	$6x - y = 0$
44.	$(4, 2)$	$x - y = 20$

In Exercises 45–48, the points represent the vertices of a triangle. (a) Draw triangle *ABC* in the coordinate plane, (b) find the altitude from vertex *B* of the triangle to side *AC*, and (c) find the area of the triangle.

45. $A = (0, 0), B = (1, 4), C = (4, 0)$

46. $A = (0, 0), B = (4, 5), C = (5, -2)$

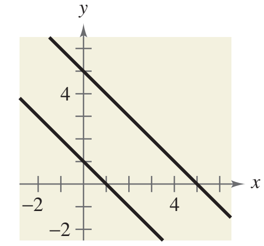
47. $A = \left(-\frac{1}{2}, \frac{1}{2}\right), B = (2, 3), C = \left(\frac{5}{2}, 0\right)$

48. $A = (-4, -5), B = (3, 10), C = (6, 12)$

In Exercises 49 and 50, find the distance between the parallel lines.

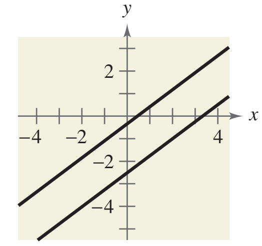
49. $x + y = 1$

$x + y = 5$

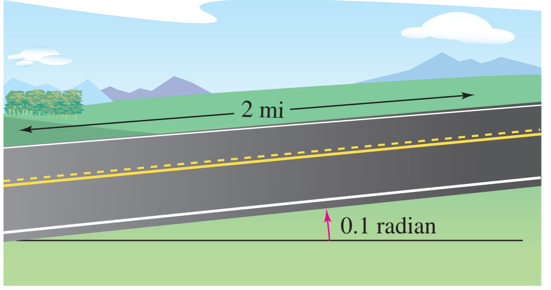


50. $3x - 4y = 1$

$3x - 4y = 10$



51. ***Road Grade*** A straight road rises with an inclination of 0.10 radian from the horizontal (see figure). Find the slope of the road and the change in elevation over a two-mile stretch of the road.



52. ***Road Grade*** A straight road rises with an inclination of 0.20 radian from the horizontal. Find the slope of the road and the change in elevation over a one-mile stretch of the road.

53. ***Pitch of a Roof*** A roof has a rise of 3 feet for every horizontal change of 5 feet (see figure). Find the inclination of the roof.

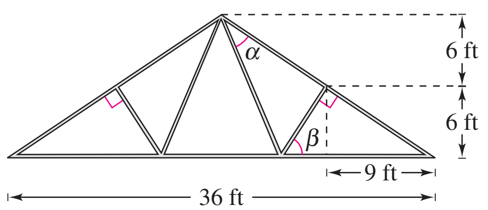
54. ***Conveyor Design*** A moving conveyor is built so that it rises 1 meter for each 3 meters of horizontal travel.

(a) Draw a diagram that gives a visual representation of the problem.

(b) Find the inclination of the conveyor.

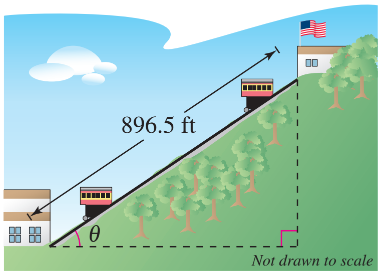
(c) The conveyor runs between two floors in a factory. The distance between the floors is 5 meters. Find the length of the conveyor.

55. ***Truss*** Find the angles α and β shown in the drawing of the roof truss.



Model It

56. ***Inclined Plane*** The Johnstown Inclined Plane in Johnstown, Pennsylvania is an inclined railway that was designed to carry people to the hilltop community of Westmont. It also proved useful in carrying people and vehicles to safety during severe floods. The railway is 896.5 feet long with a 70.9% uphill grade (see figure).



(a) Find the inclination θ of the railway.

(b) Find the change in elevation from the base to the top of the railway.

(c) Using the origin of a rectangular coordinate system as the base of the inclined plane, find the equation of the line that models the railway track.

(d) Sketch a graph of the equation you found in part (c).

Synthesis

True or False? **In Exercises 57 and 58, determine whether the statement is true or false. Justify your answer.**

57. A line that has an inclination greater than $\pi/2$ radians has a negative slope.

58. To find the angle between two lines whose angles of inclination θ_1 and θ_2 are known, substitute θ_1 and θ_2 for m_1 and m_2, respectively, in the formula for the angle between two lines.

59. ***Exploration*** Consider a line with slope m and y-intercept $(0, 4)$.

(a) Write the distance d between the origin and the line as a function of m.

(b) Graph the function in part (a).

(c) Find the slope that yields the maximum distance between the origin and the line.

(d) Find the asymptote of the graph in part (b) and interpret its meaning in the context of the problem.

60. ***Exploration*** Consider a line with slope m and y-intercept $(0, 4)$.

(a) Write the distance d between the point $(3, 1)$ and the line as a function of m.

(b) Graph the function in part (a).

(c) Find the slope that yields the maximum distance between the point and the line.

(d) Is it possible for the distance to be 0? If so, what is the slope of the line that yields a distance of 0?

(e) Find the asymptote of the graph in part (b) and interpret its meaning in the context of the problem.

Skills Review

In Exercises 61–66, find all x-intercepts and y-intercepts of the graph of the quadratic function.

61. $f(x) = (x - 7)^2$

62. $f(x) = (x + 9)^2$

63. $f(x) = (x - 5)^2 - 5$

64. $f(x) = (x + 11)^2 + 12$

65. $f(x) = x^2 - 7x - 1$

66. $f(x) = x^2 + 9x - 22$

In Exercises 67–72, write the quadratic function in standard form by completing the square. Identify the vertex of the function.

67. $f(x) = 3x^2 + 2x - 16$

68. $f(x) = 2x^2 - x - 21$

69. $f(x) = 5x^2 + 34x - 7$

70. $f(x) = -x^2 - 8x - 15$

71. $f(x) = 6x^2 - x - 12$

72. $f(x) = -8x^2 - 34x - 21$

In Exercises 73–76, graph the quadratic function.

73. $f(x) = (x - 4)^2 + 3$

74. $f(x) = 6 - (x + 1)^2$

75. $g(x) = 2x^2 - 3x + 1$

76. $g(x) = -x^2 + 6x - 8$