9 Chapter Summary

What did you learn?

Section 9.1	Review Exercises
☐ Use sequence notation to write the terms of sequences *(p. 642)*.	1–8
☐ Use factorial notation *(p. 644)*.	9–12
☐ Use summation notation to write sums *(p. 646)*.	13–20
☐ Find the sums of infinite series *(p. 647)*.	21–24
☐ Use sequences and series to model and solve real-life problems *(p. 648)*.	25, 26
Section 9.2	
☐ Recognize, write, and find the *n*th terms of arithmetic sequences *(p. 653)*.	27–40
☐ Find *n*th partial sums of arithmetic sequences *(p. 656)*.	41–46
☐ Use arithmetic sequences to model and solve real-life problems *(p. 657)*.	47, 48
Section 9.3	
☐ Recognize, write, and find the *n*th terms of geometric sequences *(p. 663)*.	49–60
☐ Find *n*th partial sums of geometric sequences *(p. 666)*.	61–70
☐ Find sums of infinite geometric series *(p. 667)*.	71–76
☐ Use geometric sequences to model and solve real-life problems *(p. 668)*.	77, 78
Section 9.4	
☐ Use mathematical induction to prove statements involving a positive integer *n* *(p. 673)*.	79–82
☐ Recognize patterns and write the *n*th term of a sequence *(p. 677)*.	83–86
☐ Find the sums of powers of integers *(p. 679)*.	87–90
☐ Find finite differences of sequences *(p. 680)*.	91–94
Section 9.5	
☐ Use the Binomial Theorem to calculate binomial coefficients *(p. 683)*.	95–98
☐ Use Pascal's Triangle to calculate binomial coefficients *(p. 685)*.	99–102
☐ Use binomial coefficients to write binomial expansions *(p. 686)*.	103–108
Section 9.6	
☐ Solve simple counting problems *(p. 691)*.	109, 110
☐ Use the Fundamental Counting Principle to solve counting problems *(p. 692)*.	111, 112
☐ Use permutations to solve counting problems *(p. 693)*.	113, 114
☐ Use combinations to solve counting problems *(p. 696)*.	115, 116
Section 9.7	
☐ Find the probabilities of events *(p. 701)*.	117, 118
☐ Find the probabilities of mutually exclusive events *(p. 705)*.	119, 120
☐ Find the probabilities of independent events *(p. 707)*.	121, 122
☐ Find the probability of the complement of an event *(p. 708)*.	123, 124

9 Review Exercises

9.1 In Exercises 1–4, write the first five terms of the sequence. (Assume that n begins with 1.)

1. $a_n = 2 + \dfrac{6}{n}$

2. $a_n = \dfrac{(-1)^n 5n}{2n - 1}$

3. $a_n = \dfrac{72}{n!}$

4. $a_n = n(n - 1)$

In Exercises 5–8, write an expression for the apparent nth term of the sequence. (Assume that n begins with 1.)

5. $-2, 2, -2, 2, -2, . . .$

6. $-1, 2, 7, 14, 23, . . .$

7. $4, 2, \frac{4}{3}, 1, \frac{4}{5}, . . .$

8. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, . . .$

In Exercises 9–12, simplify the factorial expression.

9. $5!$

10. $3! \cdot 2!$

11. $\dfrac{3! \cdot 5!}{6!}$

12. $\dfrac{7! \cdot 6!}{6! \cdot 8!}$

In Exercises 13–18, find the sum.

13. $\displaystyle\sum_{i=1}^{6} 5$

14. $\displaystyle\sum_{k=2}^{5} 4k$

15. $\displaystyle\sum_{j=1}^{4} \frac{6}{j^2}$

16. $\displaystyle\sum_{i=1}^{8} \frac{i}{i+1}$

17. $\displaystyle\sum_{k=1}^{10} 2k^3$

18. $\displaystyle\sum_{j=0}^{4} (j^2 + 1)$

In Exercises 19 and 20, use sigma notation to write the sum.

19. $\dfrac{1}{2(1)} + \dfrac{1}{2(2)} + \dfrac{1}{2(3)} + \cdot \cdot \cdot + \dfrac{1}{2(20)}$

20. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \cdot \cdot \cdot + \dfrac{9}{10}$

In Exercises 21–24, find the sum of the infinite series.

21. $\displaystyle\sum_{i=1}^{\infty} \frac{5}{10^i}$

22. $\displaystyle\sum_{i=1}^{\infty} \frac{3}{10^i}$

23. $\displaystyle\sum_{k=1}^{\infty} \frac{2}{100^k}$

24. $\displaystyle\sum_{k=2}^{\infty} \frac{9}{10^k}$

25. ***Compound Interest*** A deposit of \$10,000 is made in an account that earns 8% interest compounded monthly. The balance in the account after n months is given by

$$A_n = 10{,}000\left(1 + \frac{0.08}{12}\right)^n, \quad n = 1, 2, 3, . . .$$

(a) Write the first 10 terms of this sequence.

(b) Find the balance in this account after 10 years by finding the 120th term of the sequence.

26. ***Education*** The enrollment a_n (in thousands) in Head Start programs in the United States from 1994 to 2002 can be approximated by the model

$$a_n = 1.07n^2 + 6.1n + 693, \quad n = 4, 5, . . ., 12$$

where n is the year, with $n = 4$ corresponding to 1994. Find the terms of this finite sequence. Use a graphing utility to construct a bar graph that represents the sequence. (Source: U.S. Administration for Children and Families)

9.2 In Exercises 27–30, determine whether the sequence is arithmetic. If so, find the common difference.

27. $5, 3, 1, -1, -3, . . .$

28. $0, 1, 3, 6, 10, . . .$

29. $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, . . .$

30. $\frac{9}{9}, \frac{8}{9}, \frac{7}{9}, \frac{6}{9}, \frac{5}{9}, . . .$

In Exercises 31–34, write the first five terms of the arithmetic sequence.

31. $a_1 = 4, d = 3$

32. $a_1 = 6, d = -2$

33. $a_1 = 25, a_{k+1} = a_k + 3$

34. $a_1 = 4.2, a_{k+1} = a_k + 0.4$

In Exercises 35–40, find a formula for a_n for the arithmetic sequence.

35. $a_1 = 7, d = 12$

36. $a_1 = 25, d = -3$

37. $a_1 = y, d = 3y$

38. $a_1 = -2x, d = x$

39. $a_2 = 93, a_6 = 65$

40. $a_7 = 8, a_{13} = 6$

In Exercises 41–44, find the partial sum.

41. $\displaystyle\sum_{j=1}^{10} (2j - 3)$

42. $\displaystyle\sum_{j=1}^{8} (20 - 3j)$

43. $\displaystyle\sum_{k=1}^{11} \left(\tfrac{2}{3}k + 4\right)$

44. $\displaystyle\sum_{k=1}^{25} \left(\frac{3k+1}{4}\right)$

45. Find the sum of the first 100 positive multiples of 5.

46. Find the sum of the integers from 20 to 80 (inclusive).

47. ***Job Offer*** The starting salary for an accountant is \$34,000 with a guaranteed salary increase of \$2250 per year. Determine (a) the salary during the fifth year and (b) the total compensation through 5 full years of employment.

48. ***Baling Hay*** In the first two trips baling hay around a large field, a farmer obtains 123 bales and 112 bales, respectively. Because each round gets shorter, the farmer estimates that the same pattern will continue. Estimate the total number of bales made if the farmer takes another six trips around the field.

9.3 **In Exercises 49–52, determine whether the sequence is geometric. If so, find the common ratio.**

49. $5, 10, 20, 40, \ldots$

50. $54, -18, 6, -2, \ldots$

51. $\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{8}{3}, \ldots$

52. $\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \ldots$

In Exercises 53–56, write the first five terms of the geometric sequence.

53. $a_1 = 4, \ r = -\frac{1}{4}$

54. $a_1 = 2, \ r = 2$

55. $a_1 = 9, \ a_3 = 4$

56. $a_1 = 2, \ a_3 = 12$

In Exercises 57–60, write an expression for the nth term of the geometric sequence. Then find the 20th term of the sequence.

57. $a_1 = 16, a_2 = -8$

58. $a_3 = 6, a_4 = 1$

59. $a_1 = 100, r = 1.05$

60. $a_1 = 5, r = 0.2$

In Exercises 61–66, find the sum of the finite geometric sequence.

61. $\sum_{i=1}^{7} 2^{i-1}$

62. $\sum_{i=1}^{5} 3^{i-1}$

63. $\sum_{i=1}^{4} \left(\frac{1}{2}\right)^i$

64. $\sum_{i=1}^{6} \left(\frac{1}{3}\right)^{i-1}$

65. $\sum_{i=1}^{5} (2)^{i-1}$

66. $\sum_{i=1}^{4} 6(3)^i$

In Exercises 67–70, use a graphing utility to find the sum of the finite geometric sequence.

67. $\sum_{i=1}^{10} 10\left(\frac{3}{5}\right)^{i-1}$

68. $\sum_{i=1}^{15} 20(0.2)^{i-1}$

69. $\sum_{i=1}^{25} 100(1.06)^{i-1}$

70. $\sum_{i=1}^{20} 8\left(\frac{6}{5}\right)^{i-1}$

In Exercises 71–76, find the sum of the infinite geometric series.

71. $\sum_{i=1}^{\infty} \left(\frac{7}{8}\right)^{i-1}$

72. $\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1}$

73. $\sum_{i=1}^{\infty} (0.1)^{i-1}$

74. $\sum_{i=1}^{\infty} (0.5)^{i-1}$

75. $\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^{k-1}$

76. $\sum_{k=1}^{\infty} 1.3\left(\frac{1}{10}\right)^{k-1}$

77. ***Depreciation*** A paper manufacturer buys a machine for \$120,000. During the next 5 years, it will depreciate at a rate of 30% per year. (That is, at the end of each year the depreciated value will be 70% of what it was at the beginning of the year.)

(a) Find the formula for the nth term of a geometric sequence that gives the value of the machine t full years after it was purchased.

(b) Find the depreciated value of the machine after 5 full years.

78. ***Annuity*** You deposit \$200 in an account at the beginning of each month for 10 years. The account pays 6% compounded monthly. What will your balance be at the end of 10 years? What would the balance be if the interest were compounded continuously?

9.4 **In Exercises 79–82, use mathematical induction to prove the formula for every positive integer n.**

79. $3 + 5 + 7 + \cdots + (2n + 1) = n(n + 2)$

80. $1 + \frac{3}{2} + 2 + \frac{5}{2} + \cdots + \frac{1}{2}(n + 1) = \frac{n}{4}(n + 3)$

81. $\sum_{i=0}^{n-1} ar^i = \frac{a(1 - r^n)}{1 - r}$

82. $\sum_{k=0}^{n-1} (a + kd) = \frac{n}{2}[2a + (n - 1)d]$

In Exercises 83–86, find a formula for the sum of the first n terms of the sequence.

83. $9, 13, 17, 21, \ldots$

84. $68, 60, 52, 44, \ldots$

85. $1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}, \ldots$

86. $12, -1, \frac{1}{12}, -\frac{1}{144}, \ldots$

In Exercises 87–90, find the sum using the formulas for the sums of powers of integers.

87. $\sum_{n=1}^{30} n$

88. $\sum_{n=1}^{10} n^2$

89. $\sum_{n=1}^{7} (n^4 - n)$

90. $\sum_{n=1}^{6} (n^5 - n^2)$

In Exercises 91–94, write the first five terms of the sequence beginning with the given term. Then calculate the first and second differences of the sequence. State whether the sequence has a linear model, a quadratic model, or neither.

91. $a_1 = 5$
$a_n = a_{n-1} + 5$

92. $a_1 = -3$
$a_n = a_{n-1} - 2n$

93. $a_1 = 16$
$a_n = a_{n-1} - 1$

94. $a_0 = 0$
$a_n = n - a_{n-1}$

9.5 In Exercises 95–98, use the Binomial Theorem to calculate the binomial coefficient.

95. ${}_6C_4$ **96.** ${}_{10}C_7$

97. ${}_8C_5$ **98.** ${}_{12}C_3$

In Exercises 99–102, use Pascal's Triangle to calculate the binomial coefficient.

99. $\binom{7}{3}$ **100.** $\binom{9}{4}$

101. $\binom{8}{6}$ **102.** $\binom{5}{3}$

In Exercises 103–108, use the Binomial Theorem to expand and simplify the expression. (Remember that $i = \sqrt{-1}$.)

103. $(x + 4)^4$

104. $(x - 3)^6$

105. $(a - 3b)^5$

106. $(3x + y^2)^7$

107. $(5 + 2i)^4$

108. $(4 - 5i)^3$

9.6 109. ***Numbers in a Hat*** Slips of paper numbered 1 through 14 are placed in a hat. In how many ways can you draw two numbers with replacement that total 12?

110. ***Home Theater Systems*** A customer in an electronics store can choose one of six speaker systems, one of five DVD players, and one of six plasma televisions to design a home theater system. How many systems can be designed?

111. ***Telephone Numbers*** The same three-digit prefix is used for all of the telephone numbers in a small town. How many different telephone numbers are possible by changing only the last four digits?

112. ***Course Schedule*** A college student is preparing a course schedule for the next semester. The student may select one of three mathematics courses, one of four science courses, and one of six history courses. How many schedules are possible?

113. ***Bike Race*** There are 10 bicyclists entered in a race. In how many different ways could the top three places be decided?

114. ***Jury Selection*** A group of potential jurors has been narrowed down to 32 people. In how many ways can a jury of 12 people be selected?

115. ***Apparel*** You have eight different suits to choose from to take on a trip. How many combinations of three suits could you take on your trip?

116. ***Menu Choices*** A local sub shop offers five different breads, seven different meats, three different cheeses, and six different vegetables. Find the total number of combinations of sandwiches possible.

9.7 117. ***Apparel*** A man has five pairs of socks, of which no two pairs are the same color. He randomly selects two socks from a drawer. What is the probability that he gets a matched pair?

118. ***Bookshelf Order*** A child returns a five-volume set of books to a bookshelf. The child is not able to read, and so cannot distinguish one volume from another. What is the probability that the books are shelved in the correct order?

119. ***Students by Class*** At a particular university, the numbers of students in the four classes are broken down by percents, as shown in the table.

Class	Percent
Freshmen	31
Sophomores	26
Juniors	25
Seniors	18

A single student is picked randomly by lottery for a cash scholarship. What is the probability that the scholarship winner is

(a) a junior or senior?

(b) a freshman, sophomore, or junior?

120. ***Data Analysis*** A sample of college students, faculty, and administration were asked whether they favored a proposed increase in the annual activity fee to enhance student life on campus. The results of the study are listed in the table.

	Students	Faculty	Admin.	Total
Favor	237	37	18	292
Oppose	163	38	7	208
Total	400	75	25	500

A person is selected at random from the sample. Find each specified probability.

(a) The person is not in favor of the proposal.

(b) The person is a student.

(c) The person is a faculty member and is in favor of the proposal.

121. ***Tossing a Die*** A six-sided die is tossed three times. What is the probability of getting a 6 on each roll?

122. ***Tossing a Die*** A six-sided die is tossed six times. What is the probability that each side appears exactly once?

123. ***Drawing a Card*** You randomly select a card from a 52-card deck. What is the probability that the card is *not* a club?

124. ***Tossing a Coin*** Find the probability of obtaining at least one tail when a coin is tossed five times.

Synthesis

True or False? **In Exercises 125–129, determine whether the statement is true or false. Justify your answer.**

125. $\dfrac{(n+2)!}{n!} = (n+2)(n+1)$

126. $\displaystyle\sum_{i=1}^{5}(i^3 + 2i) = \sum_{i=1}^{5} i^3 + \sum_{i=1}^{5} 2i$

127. $\displaystyle\sum_{k=1}^{8} 3k = 3\sum_{k=1}^{8} k$

128. $\displaystyle\sum_{j=1}^{6} 2^j = \sum_{j=3}^{8} 2^{j-2}$

129. The value of ${}_nP_r$ is always greater than the value of ${}_nC_r$.

130. ***Think About It*** An infinite sequence is a function. What is the domain of the function?

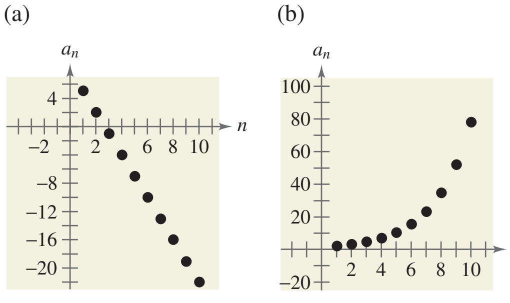
131. ***Think About It*** How do the two sequences differ?

(a) $a_n = \dfrac{(-1)^n}{n}$

(b) $a_n = \dfrac{(-1)^{n+1}}{n}$

132. ***Graphical Reasoning*** The graphs of two sequences are shown below. Identify each sequence as arithmetic or geometric. Explain your reasoning.

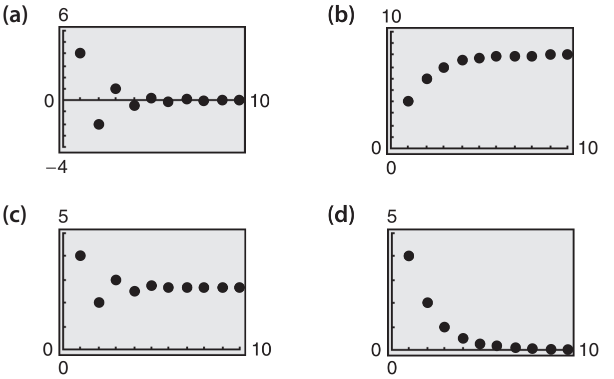
(a) (b)



133. ***Writing*** Explain what is meant by a recursion formula.

134. ***Writing*** Explain why the terms of a geometric sequence decrease when $0 < r < 1$.

Graphical Reasoning **In Exercises 135–138, match the sequence or sum of a sequence with its graph without doing any calculations. Explain your reasoning. [The graphs are labeled (a), (b), (c), and (d).]**



135. $a_n = 4\left(\frac{1}{2}\right)^{n-1}$

136. $a_n = 4\left(-\frac{1}{2}\right)^{n-1}$

137. $a_n = \displaystyle\sum_{k=1}^{n} 4\left(\frac{1}{2}\right)^{k-1}$

138. $a_n = \displaystyle\sum_{k=1}^{n} 4\left(-\frac{1}{2}\right)^{k-1}$

139. ***Population Growth*** Consider an idealized population with the characteristic that each member of the population produces one offspring at the end of every time period. If each member has a life span of three time periods and the population begins with 10 newborn members, then the following table shows the population during the first five time periods.

Age Bracket	Time Period				
	1	2	3	4	5
0–1	10	10	20	40	70
1–2		10	10	20	40
2–3			10	10	20
Total	10	20	40	70	130

The sequence for the total population has the property that

$$S_n = S_{n-1} + S_{n-2} + S_{n-3}, \quad n > 3.$$

Find the total population during the next five time periods.

140. The probability of an event must be a real number in what interval? Is the interval open or closed?

9 Chapter Test

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Write the first five terms of the sequence $a_n = \dfrac{(-1)^n}{3n + 2}$. (Assume that n begins with 1.)
2. Write an expression for the nth term of the sequence.

   $\dfrac{3}{1!}, \dfrac{4}{2!}, \dfrac{5}{3!}, \dfrac{6}{4!}, \dfrac{7}{5!}, \ldots$
3. Find the next three terms of the series. Then find the fifth partial sum of the series.

   $6 + 17 + 28 + 39 + \cdots$
4. The fifth term of an arithmetic sequence is 5.4, and the 12th term is 11.0. Find the nth term.
5. Write the first five terms of the sequence $a_n = 5(2)^{n-1}$. (Assume that n begins with 1.)

In Exercises 6–8, find the sum.

6. $\displaystyle\sum_{i=1}^{50} (2i^2 + 5)$.
7. $\displaystyle\sum_{n=1}^{7} (8n - 5)$
8. $\displaystyle\sum_{i=1}^{\infty} 4\left(\tfrac{1}{2}\right)^i$.
9. Use mathematical induction to prove the formula.

   $5 + 10 + 15 + \cdots + 5n = \dfrac{5n(n + 1)}{2}$
10. Use the Binomial Theorem to expand the expression $(x + 2y)^4$.
11. Find the coefficient of the term a^3b^5 in the expansion of $(2a - 3b)^8$.

In Exercises 12 and 13, evaluate each expression.

12. (a) ${}_9P_2$ (b) ${}_{70}P_3$
13. (a) ${}_{11}C_4$ (b) ${}_{66}C_4$
14. How many distinct license plates can be issued consisting of one letter followed by a three-digit number?
15. Eight people are going for a ride in a boat that seats eight people. The owner of the boat will drive, and only three of the remaining people are willing to ride in the two bow seats. How many seating arrangements are possible?
16. You attend a karaoke night and hope to hear your favorite song. The karaoke song book has 300 different songs (your favorite song is among the 300 songs). Assuming that the singers are equally likely to pick any song and no song is repeated, what is the probability that your favorite song is one of the 20 that you hear that night?
17. You are with seven of your friends at a party. Names of all of the 60 guests are placed in a hat and drawn randomly to award eight door prizes. Each guest is limited to one prize. What is the probability that you and your friends win all eight of the prizes?
18. The weather report calls for a 75% chance of snow. According to this report, what is the probability that it will *not* snow?

9 Cumulative Test for Chapters 7–9

Take this test to review the material from earlier chapters. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–4, solve the system by the specified method.

1. Substitution

$$\begin{cases} y = 3 - x^2 \\ 2(y - 2) = x - 1 \end{cases}$$

2. Elimination

$$\begin{cases} x + 3y = -1 \\ 2x + 4y = 0 \end{cases}$$

3. Elimination

$$\begin{cases} -2x + 4y - z = 3 \\ x - 2y + 2z = -6 \\ x - 3y - z = 1 \end{cases}$$

4. Gauss-Jordan Elimination

$$\begin{cases} x + 3y - 2z = -7 \\ -2x + y - z = -5 \\ 4x + y + z = 3 \end{cases}$$

In Exercises 5 and 6, sketch the graph of the solution set of the system of inequalities.

5. $\begin{cases} 2x + y \geq -3 \\ x - 3y \leq 2 \end{cases}$

6. $\begin{cases} x - y > 6 \\ 5x + 2y < 10 \end{cases}$

7. Sketch the region determined by the constraints. Then find the minimum and maximum values, and where they occur, of the objective function $z = 3x + 2y$, subject to the indicated constraints.

$$\begin{aligned} x + 4y &\leq 20 \\ 2x + y &\leq 12 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

8. A custom-blend bird seed is to be mixed from seed mixtures costing \$0.75 per pound and \$1.25 per pound. How many pounds of each seed mixture are used to make 200 pounds of custom-blend bird seed costing \$0.95 per pound?

9. Find the equation of the parabola $y = ax^2 + bx + c$ passing through the points (0, 4), (3, 1), and (6, 4).

$$\begin{cases} -x + 2y - z = 9 \\ 2x - y + 2z = -9 \\ 3x + 3y - 4z = 7 \end{cases}$$

SYSTEM FOR 10 AND 11

In Exercises 10 and 11, use the system of equations at the left.

10. Write the augmented matrix corresponding to the system of equations.

11. Solve the system using the matrix found in Exercise 10 and Gauss-Jordan elimination.

In Exercises 12–15, use the following matrices to find each of the following, if possible.

$$A = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$$

12. $A + B$

13. $-2B$

14. $A - 2B$

15. AB

$$\begin{bmatrix} 8 & 0 & -5 \\ 1 & 3 & -1 \\ -2 & 6 & 4 \end{bmatrix}$$

MATRIX FOR 16

16. Find the determinant of the matrix at the left.

17. Find the inverse of the matrix (if it exists): $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$.

$$\begin{array}{ccc} & & \text{Gym shoes} & \text{Jogging shoes} & \text{Walking shoes} \\ \text{Age group} & \begin{cases} 14-17 \\ 18-24 \\ 25-34 \end{cases} & \multicolumn{3}{c}{\begin{bmatrix} 0.09 & 0.09 & 0.03 \\ 0.06 & 0.10 & 0.05 \\ 0.12 & 0.25 & 0.12 \end{bmatrix}} \end{array}$$

MATRIX FOR 18

18. The percents (by age group) of the total amounts spent on three types of footwear in a recent year are shown in the matrix. The total amounts (in millions) spent by each age group on the three types of footwear were \$442.20 (14–17 age group), \$466.57(18–24 age group), and \$1088.09 (25–34 age group). How many dollars worth of gym shoes, jogging shoes, and walking shoes were sold that year? (Source: National Sporting Goods Association)

In Exercises 19 and 20, use Cramer's Rule to solve the system of equations.

19. $\begin{cases} 8x - 3y = -52 \\ 3x + 5y = 5 \end{cases}$

20. $\begin{cases} 5x + 4y + 3z = 7 \\ -3x - 8y + 7z = -9 \\ 7x - 5y - 6z = -53 \end{cases}$

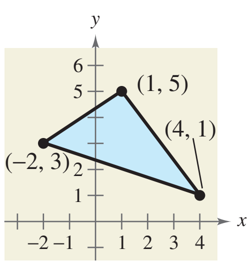


FIGURE FOR 21

21. Find the area of the triangle shown in the figure.

22. Write the first five terms of the sequence $a_n = \dfrac{(-1)^{n+1}}{2n + 3}$ (assume that n begins with 1).

23. Write an expression for the nth term of the sequence.

$\dfrac{2!}{4}, \dfrac{3!}{5}, \dfrac{4!}{6}, \dfrac{5!}{7}, \dfrac{6!}{8}, \ldots$

24. Find the sum of the first 20 terms of the arithmetic sequence $8, 12, 16, 20, \ldots$.

25. The sixth term of an arithmetic sequence is 20.6, and the ninth term is 30.2.

(a) Find the 20th term.

(b) Find the nth term.

26. Write the first five terms of the sequence $a_n = 3(2)^{n-1}$ (assume that n begins with 1).

27. Find the sum: $\displaystyle\sum_{i=0}^{\infty} 1.3\left(\tfrac{1}{10}\right)^{i-1}$.

28. Use mathematical induction to prove the formula

$3 + 7 + 11 + 15 + \cdot \cdot \cdot + (4n - 1) = n(2n + 1)$.

29. Use the Binomial Theorem to expand and simplify $(z - 3)^4$.

In Exercises 30–33, evaluate the expression.

30. ${}_7P_3$ **31.** ${}_{25}P_2$ **32.** $\dbinom{8}{4}$ **33.** ${}_{10}C_3$

In Exercises 34 and 35, find the number of distinguishable permutations of the group of letters.

34. B, A, S, K, E, T, B, A, L, L **35.** A, N, T, A, R, C, T, I, C, A

36. A personnel manager at a department store has 10 applicants to fill three different sales positions. In how many ways can this be done, assuming that all the applicants are qualified for any of the three positions?

37. On a game show, the digits 3, 4, and 5 must be arranged in the proper order to form the price of an appliance. If the digits are arranged correctly, the contestant wins the appliance. What is the probability of winning if the contestant knows that the price is at least \$400?

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Proofs in Mathematics

Properties of Sums (p. 647)

1. $\sum_{i=1}^{n} c = cn$, $\quad c$ is a constant.

2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, $\quad c$ is a constant.

3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Proof

Each of these properties follows directly from the properties of real numbers.

1. $\sum_{i=1}^{n} c = c + c + c + \cdot \cdot \cdot + c = cn \qquad n$ terms

The Distributive Property is used in the proof of Property 2.

2. $$\begin{aligned}\sum_{i=1}^{n} ca_i &= ca_1 + ca_2 + ca_3 + \cdot \cdot \cdot + ca_n\\ &= c(a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) = c\sum_{i=1}^{n} a_i\end{aligned}$$

The proof of Property 3 uses the Commutative and Associative Properties of Addition.

3. $$\begin{aligned}\sum_{i=1}^{n} (a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdot \cdot \cdot + (a_n + b_n)\\ &= (a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) + (b_1 + b_2 + b_3 + \cdot \cdot \cdot + b_n)\\ &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\end{aligned}$$

The proof of Property 4 uses the Commutative and Associative Properties of Addition and the Distributive Property.

4. $$\begin{aligned}\sum_{i=1}^{n} (a_i - b_i) &= (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + \cdot \cdot \cdot + (a_n - b_n)\\ &= (a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) + (-b_1 - b_2 - b_3 - \cdot \cdot \cdot - b_n)\\ &= (a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_n) - (b_1 + b_2 + b_3 + \cdot \cdot \cdot + b_n)\\ &= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\end{aligned}$$

Infinite Series

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity; throughout the next quarter of the interval at double the intensity; throughout the following eighth at triple the intensity and so ad infinitum; The average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdot \cdot \cdot + \frac{n}{2^n} + \cdot \cdot \cdot$$

is 2.

The Sum of a Finite Arithmetic Sequence (p. 656)

The sum of a finite arithmetic sequence with n terms is

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Proof

Begin by generating the terms of the arithmetic sequence in two ways. In the first way, repeatedly add d to the first term to obtain

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdot \cdot \cdot + a_{n-2} + a_{n-1} + a_n \\ &= a_1 + [a_1 + d] + [a_1 + 2d] + \cdot \cdot \cdot + [a_1 + (n-1)d]. \end{aligned}$$

In the second way, repeatedly subtract d from the nth term to obtain

$$\begin{aligned} S_n &= a_n + a_{n-1} + a_{n-2} + \cdot \cdot \cdot + a_3 + a_2 + a_1 \\ &= a_n + [a_n - d] + [a_n - 2d] + \cdot \cdot \cdot + [a_n - (n-1)d]. \end{aligned}$$

If you add these two versions of S_n, the multiples of d subtract out and you obtain

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdot \cdot \cdot + (a_1 + a_n) \quad n \text{ terms}$$

$$2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n}{2}(a_1 + a_n).$$

The Sum of a Finite Geometric Sequence (p. 666)

The sum of the finite geometric sequence

$$a_1,\ a_1 r,\ a_1 r^2,\ a_1 r^3,\ a_1 r^4,\ .\ .\ .\ ,\ a_1 r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right)$.

Proof

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdot \cdot \cdot + a_1 r^{n-2} + a_1 r^{n-1}$$

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \cdot \cdot \cdot + a_1 r^{n-1} + a_1 r^n \qquad \text{Multiply by } r.$$

Subtracting the second equation from the first yields

$$S_n - rS_n = a_1 - a_1 r^n.$$

So, $S_n(1 - r) = a_1(1 - r^n)$, and, because $r \neq 1$, you have $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$.

The Binomial Theorem (p. 683)

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdot \cdot \cdot + {}_nC_r\, x^{n-r}y^r + \cdot \cdot \cdot + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

Proof

The Binomial Theorem can be proved quite nicely using mathematical induction. The steps are straightforward but look a little messy, so only an outline of the proof is presented.

1. If $n = 1$, you have $(x + y)^1 = x^1 + y^1 = {}_1C_0x + {}_1C_1y$, and the formula is valid.

2. Assuming that the formula is true for $n = k$, the coefficient of $x^{k-r}y^r$ is

$${}_kC_r = \frac{k!}{(k-r)!r!} = \frac{k(k-1)(k-2) \cdot \cdot \cdot (k-r+1)}{r!}.$$

To show that the formula is true for $n = k + 1$, look at the coefficient of $x^{k+1-r}y^r$ in the expansion of

$$(x + y)^{k+1} = (x + y)^k(x + y).$$

From the right-hand side, you can determine that the term involving $x^{k+1-r}y^r$ is the sum of two products.

$$\begin{aligned}
&({}_kC_r x^{k-r}y^r)(x) + ({}_kC_{r-1}x^{k+1-r}y^{r-1})(y)\\
&= \left[\frac{k!}{(k-r)!r!} + \frac{k!}{(k+1-r)!(r-1)!}\right]x^{k+1-r}y^r\\
&= \left[\frac{(k+1-r)k!}{(k+1-r)!r!} + \frac{k!r}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= \left[\frac{k!(k+1-r+r)}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= \left[\frac{(k+1)!}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= {}_{k+1}C_r x^{k+1-r}y^r
\end{aligned}$$

So, by mathematical induction, the Binomial Theorem is valid for all positive integers n.

P.S. Problem Solving

This collection of thought-provoking and challenging exercises further explores and expands upon concepts learned in this chapter.

1. Let $x_0 = 1$ and consider the sequence x_n given by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad n = 1, 2, \ldots$$

Use a graphing utility to compute the first 10 terms of the sequence and make a conjecture about the value of x_n as n approaches infinity.

2. Consider the sequence

$$a_n = \frac{n+1}{n^2+1}.$$

(a) Use a graphing utility to graph the first 10 terms of the sequence.

(b) Use the graph from part (a) to estimate the value of a_n as n approaches infinity.

(c) Complete the table.

n	1	10	100	1000	10,000
a_n					

(d) Use the table from part (c) to determine (if possible) the value of a_n as n approaches infinity.

3. Consider the sequence

$$a_n = 3 + (-1)^n.$$

(a) Use a graphing utility to graph the first 10 terms of the sequence.

(b) Use the graph from part (a) to describe the behavior of the graph of the sequence.

(c) Complete the table.

n	1	10	101	1000	10,001
a_n					

(d) Use the table from part (c) to determine (if possible) the value of a_n as n approaches infinity.

4. The following operations are performed on each term of an arithmetic sequence. Determine if the resulting sequence is arithmetic, and if so, state the common difference.

(a) A constant C is added to each term.

(b) Each term is multiplied by a nonzero constant C.

(c) Each term is squared.

5. The following sequence of perfect squares is not arithmetic.

$$1, 4, 9, 16, 25, 36, 49, 64, 81, \ldots$$

However, you can form a related sequence that is arithmetic by finding the differences of consecutive terms.

(a) Write the first eight terms of the related arithmetic sequence described above. What is the nth term of this sequence?

(b) Describe how you can find an arithmetic sequence that is related to the following sequence of perfect cubes.

$$1, 8, 27, 64, 125, 216, 343, 512, 729, \ldots$$

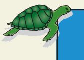
(c) Write the first seven terms of the related sequence in part (b) and find the nth term of the sequence.

(d) Describe how you can find the arithmetic sequence that is related to the following sequence of perfect fourth powers.

$$1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, \ldots$$

(e) Write the first six terms of the related sequence in part (d) and find the nth term of the sequence.

6. Can the Greek hero Achilles, running at 20 feet per second, ever catch a tortoise, starting 20 feet ahead of Achilles and running at 10 feet per second? The Greek mathematician Zeno said no. When Achilles runs 20 feet, the tortoise will be 10 feet ahead. Then, when Achilles runs 10 feet, the tortoise will be 5 feet ahead. Achilles will keep cutting the distance in half but will never catch the tortoise. The table shows Zeno's reasoning. From the table you can see that both the distances and the times required to achieve them form infinite geometric series. Using the table, show that both series have finite sums. What do these sums represent?



Distance (in feet)	Time (in seconds)
20	1
10	0.5
5	0.25
2.5	0.125
1.25	0.0625
0.625	0.03125

7. Recall that a *fractal* is a geometric figure that consists of a pattern that is repeated infinitely on a smaller and smaller scale. A well-known fractal is called the *Sierpinski Triangle*. In the first stage, the midpoints of the three sides are used to create the vertices of a new triangle, which is then removed, leaving three triangles. The first three stages are shown on the next page. Note that each remaining triangle is similar to the original triangle. Assume that the length of each side of the original triangle is one unit.

Write a formula that describes the side length of the triangles that will be generated in the nth stage. Write a formula for the area of the triangles that will be generated in the nth stage.

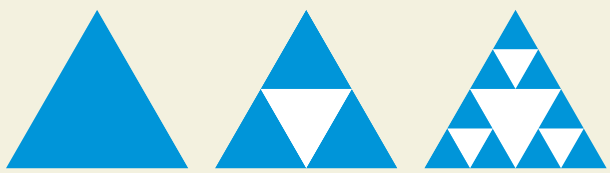


FIGURE FOR 7

8. You can define a sequence using a piecewise formula. The following is an example of a piecewise-defined sequence.

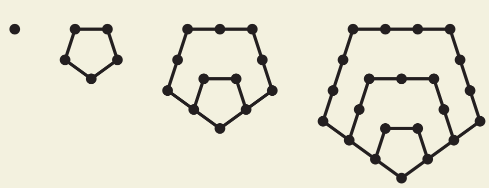
$$a_1 = 7, \quad a_n = \begin{cases} \dfrac{a_{n-1}}{2}, & \text{if } a_{n-1} \text{ is even} \\ 3a_{n-1} + 1, & \text{if } a_{n-1} \text{ is odd} \end{cases}$$

(a) Write the first 10 terms of the sequence.

(b) Choose three different values for a_1 (other than $a_1 = 7$). For each value of a_1, find the first 10 terms of the sequence. What conclusions can you make about the behavior of this sequence?

9. The numbers 1, 5, 12, 22, 35, 51, . . . are called pentagonal numbers because they represent the numbers of dots used to make pentagons, as shown below. Use mathematical induction to prove that the nth pentagonal number P_n is given by

$$P_n = \frac{n(3n-1)}{2}.$$



10. What conclusion can be drawn from the following information about the sequence of statements P_n?

(a) P_3 is true and P_k implies P_{k+1}.

(b) $P_1, P_2, P_3, \ldots, P_{50}$ are all true.

(c) P_1, P_2, and P_3 are all true, but the truth of P_k does not imply that P_{k+1} is true.

(d) P_2 is true and P_{2k} implies P_{2k+2}.

11. Let $f_1, f_2, \ldots, f_n, \ldots$ be the Fibonacci sequence.

(a) Use mathematical induction to prove that

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

(b) Find the sum of the first 20 terms of the Fibonacci sequence.

12. The odds in favor of an event occurring are the ratio of the probability that the event will occur to the probability that the event will not occur. The reciprocal of this ratio represents the odds against the event occurring.

(a) Six marbles in a bag are red. The odds against choosing a red marble are 4 to 1. How many marbles are in the bag?

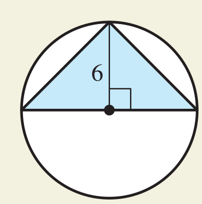
(b) A bag contains three blue marbles and seven yellow marbles. What are the odds in favor of choosing a blue marble? What are the odds against choosing a blue marble?

(c) Write a formula for converting the odds in favor of an event to the probability of the event.

(d) Write a formula for converting the probability of an event to the odds in favor of the event.

13. You are taking a test that contains only multiple choice questions (there are five choices for each question). You are on the last question and you know that the answer is not B or D, but you are not sure about answers A, C, and E. What is the probability that you will get the right answer if you take a guess?

14. A dart is thrown at the circular target shown below. The dart is equally likely to hit any point inside the target. What is the probability that it hits the region outside the triangle?



15. An event A has n possible outcomes, which have the values $x_1, x_2, \ldots, x_n$. The probabilities of the n outcomes occurring are $p_1, p_2, \ldots, p_n$. The **expected value** V of an event A is the sum of the products of the outcomes' probabilities and their values, $V = p_1x_1 + p_2x_2 + \cdots + p_nx_n$.

(a) To win California's Super Lotto Plus game, you must match five different numbers chosen from the numbers 1 to 47, plus one Mega number chosen from the numbers 1 to 27. You purchase a ticket for \$1. If the jackpot for the next drawing is \$12,000,000, what is the expected value for the ticket?

(b) You are playing a dice game in which you need to score 60 points to win. On each turn, you roll two six-sided dice. Your score for the turn is 0 if the dice do not show the same number, and the product of the numbers on the dice if they do show the same number. What is the expected value for each turn? How many turns will it take on average to score 60 points?