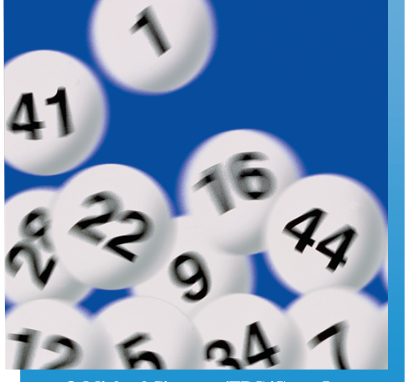
9.6 Counting Principles

What you should learn

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.

Why you should learn it

You can use counting principles to solve counting problems that occur in real life. For instance, in Exercise 65 on page 700, you are asked to use counting principles to determine the number of possible ways of selecting the winning numbers in the Powerball lottery.



Simple Counting Problems

This section and Section 9.7 present a brief introduction to some of the basic counting principles and their application to probability. In Section 9.7, you will see that much of probability has to do with counting the number of ways an event can occur. The following two examples describe simple counting problems.

Example 1 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box, its number is written down, and the piece of paper is *replaced in the box*. Then, a second piece of paper is drawn from the box, and its number is written down. Finally, the two numbers are added together. How many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the different ways that a sum of 12 can be obtained using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

From this list, you can see that a sum of 12 can occur in five different ways.

CHECKPOINT Now try Exercise 5.

Example 2 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. Two pieces of paper are drawn from the box *at the same time*, and the numbers on the pieces of paper are written down and totaled. How many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the different ways that a sum of 12 can be obtained *using two different numbers* from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can be obtained in four different ways.

CHECKPOINT Now try Exercise 7.

The difference between the counting problems in Examples 1 and 2 can be described by saying that the random selection in Example 1 occurs **with replacement,** whereas the random selection in Example 2 occurs **without replacement,** which eliminates the possibility of choosing two 6's.

You may want to consider opening class by asking students if they can predict which offers more choices for license plates: (a) a plate with three different letters of the alphabet in any order or (b) a plate with four different nonzero digits in any order. You may want to end class by verifying the answers: (a) offers $26 \cdot 25 \cdot 24 = 15{,}600$ choices and (b) offers $9 \cdot 8 \cdot 7 \cdot 6 = 3024$ choices.

The Fundamental Counting Principle

Examples 1 and 2 describe simple counting problems in which you can *list* each possible way that an event can occur. When it is possible, this is always the best way to solve a counting problem. However, some events can occur in so many different ways that it is not feasible to write out the entire list. In such cases, you must rely on formulas and counting principles. The most important of these is the **Fundamental Counting Principle.**

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1, E_2, and E_3 can occur is $m_1 \cdot m_2 \cdot m_3$.

Example 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

Solution

There are two events in this situation. The first event is the choice of the first letter, and the second event is the choice of the second letter. Because the English alphabet contains 26 letters, it follows that the number of two-letter pairs is $26 \cdot 26 = 676$.

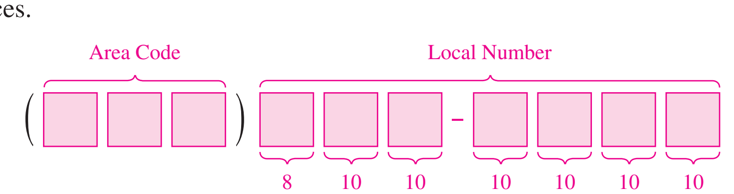
CHECKPOINT Now try Exercise 13.

Example 4 Using the Fundamental Counting Principle

Telephone numbers in the United States currently have 10 digits. The first three are the *area code* and the next seven are the *local telephone number*. How many different telephone numbers are possible within each area code? (Note that at this time, a local telephone number cannot begin with 0 or 1.)

Solution

Because the first digit of a local telephone number cannot be 0 or 1, there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of local telephone numbers that are possible *within* each area code is $8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000$.

CHECKPOINT Now try Exercise 19.

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example 5 Finding the Number of Permutations of n Elements

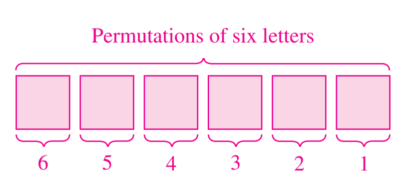
How many permutations are possible for the letters A, B, C, D, E, and F?

Solution

Consider the following reasoning.

First position: Any of the *six* letters
Second position: Any of the remaining *five* letters
Third position: Any of the remaining *four* letters
Fourth position: Any of the remaining *three* letters
Fifth position: Any of the remaining *two* letters
Sixth position: The *one* remaining letter

So, the numbers of choices for the six positions are as follows.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 720.$$

CHECKPOINT Now try Exercise 39.

Number of Permutations of n Elements

The number of permutations of n elements is

$$n \cdot (n - 1) \cdot \cdot \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n elements can be ordered.

Additional Examples

a. How many permutations are possible for the numbers 0, 1, 2, and 3?
Solution
$4! = 24$

b. Write out the possible permutations of the letters A, B, and C.
Solution
ABC, ACB, BAC, BCA, CAB, CBA

Vaughn Youtz/Newsmakers/Getty Images

Eleven thoroughbred racehorses hold the title of Triple Crown winner for winning the Kentucky Derby, the Preakness, and the Belmont Stakes in the same year. Forty-nine horses have won two out of the three races.

Example 6 Counting Horse Race Finishes

Eight horses are running in a race. In how many different ways can these horses come in first, second, and third? (Assume that there are no ties.)

Solution

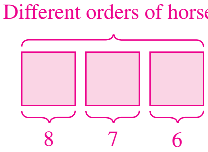
Here are the different possibilities.

Win (first position): *Eight* choices

Place (second position): *Seven* choices

Show (third position): *Six* choices

Using the Fundamental Counting Principle, multiply these three numbers together to obtain the following.



So, there are $8 \cdot 7 \cdot 6 = 336$ different orders.

CHECKPOINT Now try Exercise 43.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you might want to choose and order r elements out of a collection of n elements. Such an ordering is called a **permutation of *n* elements taken *r* at a time.**

Technology

Most graphing calculators are programmed to evaluate ${}_nP_r$. Consult the user's guide for your calculator and then evaluate ${}_8P_5$. You should get an answer of 6720.

Permutations of *n* Elements Taken *r* at a Time

The number of permutations of n elements taken r at a time is

$$\begin{aligned} {}_nP_r &= \frac{n!}{(n-r)!} \\ &= n(n-1)(n-2) \cdot \cdot \cdot (n-r+1). \end{aligned}$$

Using this formula, you can rework Example 6 to find that the number of permutations of eight horses taken three at a time is

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} \\ &= 336 \end{aligned}$$

which is the same answer obtained in the example.

Remember that for permutations, order is important. So, if you are looking at the possible permutations of the letters A, B, C, and D taken three at a time, the permutations (A, B, D) and (B, A, D) are counted as different because the *order* of the elements is different.

Suppose, however, that you are asked to find the possible permutations of the letters A, A, B, and C. The total number of permutations of the four letters would be ${}_4P_4 = 4!$. However, not all of these arrangements would be *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdot \cdot \cdot + n_k.$$

Then the number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdot \cdot \cdot \cdot n_k!}.$$

Example 7 Distinguishable Permutations

In how many distinguishable ways can the letters in BANANA be written?

Solution

This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways the letters can be written is

$$\begin{aligned}\frac{n!}{n_1! \cdot n_2! \cdot n_3!} &= \frac{6!}{3! \cdot 2! \cdot 1!}\\ &= \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!}\\ &= 60.\end{aligned}$$

The 60 different distinguishable permutations are as follows.

Additional Example

In how many different ways can the letters in INDIANA be written?

Solution

630

AAABNN	AAANBN	AAANNB	AABANN	AABNAN	AABNNA
AANABN	AANANB	AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN	ABNANA	ABNNAA
ANAABN	ANAANB	ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB	ANNABA	ANNBAA
BAAANN	BAANAN	BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA	NAAABN	NAAANB
NAABAN	NAABNA	NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA	NBAAAN	NBAANA
NBANAA	NBNAAA	NNAAAB	NNAABA	NNABAA	NNBAAA

✓CHECKPOINT Now try Exercise 45.

Combinations

When you count the number of possible permutations of a set of elements, *order* is important. As a final topic in this section, you will look at a method of selecting subsets of a larger set in which order is *not* important. Such subsets are called **combinations of *n* elements taken *r* at a time.** For instance, the combinations

$$\{A, B, C\} \qquad \text{and} \qquad \{B, A, C\}$$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important. So, you would count only one of the two sets. A common example of how a combination occurs is a card game in which the player is free to reorder the cards after they have been dealt.

Careful attention to detail and numerous examples will help students understand when to count different orders and when not to. You may want to discuss the misnaming of the "combination lock" according to the definitions presented in this section.

Example 8 Combinations of *n* Elements Taken *r* at a Time

In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)

Solution

The following subsets represent the different combinations of three letters that can be chosen from the five letters.

$$\begin{array}{ll} \{A, B, C\} & \{A, B, D\} \\ \{A, B, E\} & \{A, C, D\} \\ \{A, C, E\} & \{A, D, E\} \\ \{B, C, D\} & \{B, C, E\} \\ \{B, D, E\} & \{C, D, E\} \end{array}$$

From this list, you can conclude that there are 10 different ways that three letters can be chosen from five letters.

CHECKPOINT Now try Exercise 55.

Combinations of *n* Elements Taken *r* at a Time

The number of combinations of n elements taken r at a time is

$${}_nC_r = \frac{n!}{(n-r)!r!} \text{ which is equivalent to } {}_nC_r = \frac{{}_nP_r}{r!}.$$

Note that the formula for ${}_nC_r$ is the same one given for binomial coefficients. To see how this formula is used, solve the counting problem in Example 8. In that problem, you are asked to find the number of combinations of five elements taken three at a time. So, $n = 5$, $r = 3$, and the number of combinations is

$${}_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{2} \cdot 1 \cdot \cancel{3!}} = 10$$

which is the same answer obtained in Example 8.

A ♥	A ♦	A ♣	A ♠
2 ♥	2 ♦	2 ♣	2 ♠
3 ♥	3 ♦	3 ♣	3 ♠
4 ♥	4 ♦	4 ♣	4 ♠
5 ♥	5 ♦	5 ♣	5 ♠
6 ♥	6 ♦	6 ♣	6 ♠
7 ♥	7 ♦	7 ♣	7 ♠
8 ♥	8 ♦	8 ♣	8 ♠
9 ♥	9 ♦	9 ♣	9 ♠
10 ♥	10 ♦	10 ♣	10 ♠
J ♥	J ♦	J ♣	J ♠
Q ♥	Q ♦	Q ♣	Q ♠
K ♥	K ♦	K ♣	K ♠

FIGURE 9.7 *Standard deck of playing cards*

Example 9 Counting Card Hands

A standard poker hand consists of five cards dealt from a deck of 52 (see Figure 9.7). How many different poker hands are possible? (After the cards are dealt, the player may reorder them, and so order is not important.)

Solution

You can find the number of different poker hands by using the formula for the number of combinations of 52 elements taken five at a time, as follows.

$$\begin{aligned} {}_{52}C_5 &= \frac{52!}{(52-5)!5!} \\ &= \frac{52!}{47!5!} \\ &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} \\ &= 2{,}598{,}960 \end{aligned}$$

CHECKPOINT Now try Exercise 63.

Example 10 Forming a Team

You are forming a 12-member swim team from 10 girls and 15 boys. The team must consist of five girls and seven boys. How many different 12-member teams are possible?

Solution

There are ${}_{10}C_5$ ways of choosing five girls. The are ${}_{15}C_7$ ways of choosing seven boys. By the Fundamental Counting Principal, there are ${}_{10}C_5 \cdot {}_{15}C_7$ ways of choosing five girls and seven boys.

$$\begin{aligned} {}_{10}C_5 \cdot {}_{15}C_7 &= \frac{10!}{5! \cdot 5!} \cdot \frac{15!}{8! \cdot 7!} \\ &= 252 \cdot 6435 \\ &= 1{,}621{,}620 \end{aligned}$$

So, there are 1,621,620 12-member swim teams possible.

CHECKPOINT Now try Exercise 65.

When solving problems involving counting principles, you need to be able to distinguish among the various counting principles in order to determine which is necessary to solve the problem correctly. To do this, ask yourself the following questions.

1. Is the order of the elements important? *Permutation*
2. Are the chosen elements a subset of a larger set in which order is not important? *Combination*
3. Does the problem involve two or more separate events? *Fundamental Counting Principle*

Activities

1. Evaluate (a) ${}_6P_2$ and (b) ${}_6C_2$.
   Answers: (a) 30 and (b) 15
2. A local building supply company is hiring extra summer help. They need four additional employees to work outside in the lumber yard and three more to work inside the store. In how many ways can these positions be filled if there are 10 applicants for outside work and 5 for inside work?
   Answer: 2100
3. In how many distinguishable ways can the letters C A L C U L U S be written?
   Answer: 5040

9.6 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The ________ ________ ________ states that if there are m_1 ways for one event to occur and m_2 ways for a second event to occur, there are $m_1 \cdot m_2$ ways for both events to occur.
2. An ordering of n elements is called a ________ of the elements.
3. The number of permutations of n elements taken r at a time is given by the formula ________.
4. The number of ________ ________ of n objects is given by $\dfrac{n!}{n_1!n_2!n_3! \cdot \cdot \cdot n_k!}$.
5. When selecting subsets of a larger set in which order is not important, you are finding the number of ________ of n elements taken r at a time.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

Random Selection **In Exercises 1–8, determine the number of ways a computer can randomly generate one or more such integers from 1 through 12.**

1. An odd integer

2. An even integer

3. A prime integer

4. An integer that is greater than 9

5. An integer that is divisible by 4

6. An integer that is divisible by 3

7. Two *distinct* integers whose sum is 9

8. Two *distinct* integers whose sum is 8

9. ***Entertainment Systems*** A customer can choose one of three amplifiers, one of two compact disc players, and one of five speaker models for an entertainment system. Determine the number of possible system configurations.

10. ***Job Applicants*** A college needs two additional faculty members: a chemist and a statistician. In how many ways can these positions be filled if there are five applicants for the chemistry position and three applicants for the statistics position?

11. ***Course Schedule*** A college student is preparing a course schedule for the next semester. The student may select one of two mathematics courses, one of three science courses, and one of five courses from the social sciences and humanities. How many schedules are possible?

12. ***Aircraft Boarding*** Eight people are boarding an aircraft. Two have tickets for first class and board before those in the economy class. In how many ways can the eight people board the aircraft?

13. ***True-False Exam*** In how many ways can a six-question true-false exam be answered? (Assume that no questions are omitted.)

14. ***True-False Exam*** In how many ways can a 12-question true-false exam be answered? (Assume that no questions are omitted.)

15. ***License Plate Numbers*** In the state of Pennsylvania, each standard automobile license plate number consists of three letters followed by a four-digit number. How many distinct license plate numbers can be formed in Pennsylvania?

16. ***License Plate Numbers*** In a certain state, each automobile license plate number consists of two letters followed by a four-digit number. To avoid confusion between “O” and “zero” and between “I” and “one,” the letters “O” and “I” are not used. How many distinct license plate numbers can be formed in this state?

17. ***Three-Digit Numbers*** How many three-digit numbers can be formed under each condition?

(a) The leading digit cannot be zero.

(b) The leading digit cannot be zero and no repetition of digits is allowed.

(c) The leading digit cannot be zero and the number must be a multiple of 5.

(d) The number is at least 400.

18. ***Four-Digit Numbers*** How many four-digit numbers can be formed under each condition?

(a) The leading digit cannot be zero.

(b) The leading digit cannot be zero and no repetition of digits is allowed.

(c) The leading digit cannot be zero and the number must be less than 5000.

(d) The leading digit cannot be zero and the number must be even.

19. ***Combination Lock*** A combination lock will open when the right choice of three numbers (from 1 to 40, inclusive) is selected. How many different lock combinations are possible?

20. ***Combination Lock*** A combination lock will open when the right choice of three numbers (from 1 to 50, inclusive) is selected. How many different lock combinations are possible?

21. ***Concert Seats*** Four couples have reserved seats in a row for a concert. In how many different ways can they be seated if

(a) there are no seating restrictions?

(b) the two members of each couple wish to sit together?

22. ***Single File*** In how many orders can four girls and four boys walk through a doorway single file if

(a) there are no restrictions?

(b) the girls walk through before the boys?

In Exercises 23–28, evaluate $_nP_r$.

23. $_4P_4$ **24.** $_5P_5$

25. $_8P_3$ **26.** $_{20}P_2$

27. $_5P_4$ **28.** $_7P_4$

In Exercises 29 and 30, solve for *n*.

29. $14 \cdot {}_nP_3 = {}_{n+2}P_4$ **30.** ${}_nP_5 = 18 \cdot {}_{n-2}P_4$

In Exercises 31–36, evaluate using a graphing utility.

31. $_{20}P_5$ **32.** $_{100}P_5$

33. $_{100}P_3$ **34.** $_{10}P_8$

35. $_{20}C_5$ **36.** $_{10}C_7$

37. ***Posing for a Photograph*** In how many ways can five children posing for a photograph line up in a row?

38. ***Riding in a Car*** In how many ways can six people sit in a six-passenger car?

39. ***Choosing Officers*** From a pool of 12 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. In how many different ways can the offices be filled?

40. ***Assembly Line Production*** There are four processes involved in assembling a product, and these processes can be performed in any order. The management wants to test each order to determine which is the least time-consuming. How many different orders will have to be tested?

In Exercises 41–44, find the number of distinguishable permutations of the group of letters.

41. A, A, G, E, E, E, M **42.** B, B, B, T, T, T, T, T

43. A, L, G, E, B, R, A **44.** M, I, S, S, I, S, S, I, P, P, I

45. Write all permutations of the letters A, B, C, and D.

46. Write all permutations of the letters A, B, C, and D if the letters B and C must remain between the letters A and D.

47. ***Batting Order*** A baseball coach is creating a nine-player batting order by selecting from a team of 15 players. How many different batting orders are possible?

48. ***Athletics*** Six sprinters have qualified for the finals in the 100-meter dash at the NCAA national track meet. In how many ways can the sprinters come in first, second, and third? (Assume there are no ties.)

49. ***Jury Selection*** From a group of 40 people, a jury of 12 people is to be selected. In how many different ways can the jury be selected?

50. ***Committee Members*** As of January 2005, the U.S. Senate Committee on Indian Affairs had 14 members. Assuming party affiliation was not a factor in selection, how many different committees were possible from the 100 U.S. senators?

51. Write all possible selections of two letters that can be formed from the letters A, B, C, D, E, and F. (The order of the two letters is not important.)

52. ***Forming an Experimental Group*** In order to conduct an experiment, five students are randomly selected from a class of 20. How many different groups of five students are possible?

53. ***Lottery Choices*** In the Massachusetts Mass Cash game, a player chooses five distinct numbers from 1 to 35. In how many ways can a player select the five numbers?

54. ***Lottery Choices*** In the Louisiana Lotto game, a player chooses six distinct numbers from 1 to 40. In how many ways can a player select the six numbers?

55. ***Defective Units*** A shipment of 10 microwave ovens contains three defective units. In how many ways can a vending company purchase four of these units and receive (a) all good units, (b) two good units, and (c) at least two good units?

56. ***Interpersonal Relationships*** The complexity of interpersonal relationships increases dramatically as the size of a group increases. Determine the numbers of different two-person relationships in groups of people of sizes (a) 3, (b) 8, (c) 12, and (d) 20.

57. ***Poker Hand*** You are dealt five cards from an ordinary deck of 52 playing cards. In how many ways can you get (a) a full house and (b) a five-card combination containing two jacks and three aces? (A full house consists of three of one kind and two of another. For example, A-A-A-5-5 and K-K-K-10-10 are full houses.)

58. ***Job Applicants*** A toy manufacturer interviews eight people for four openings in the research and development department of the company. Three of the eight people are women. If all eight are qualified, in how many ways can the employer fill the four positions if (a) the selection is random and (b) exactly two selections are women?

59. ***Forming a Committee*** A six-member research committee at a local college is to be formed having one administrator, three faculty members, and two students. There are seven administrators, 12 faculty members, and 20 students in contention for the committee. How many six-member committees are possible?

60. ***Law Enforcement*** A police department uses computer imaging to create digital photographs of alleged perpetrators from eyewitness accounts. One software package contains 195 hairlines, 99 sets of eyes and eyebrows, 89 noses, 105 mouths, and 74 chins and cheek structures.

(a) Find the possible number of different faces that the software could create.

(b) A eyewitness can clearly recall the hairline and eyes and eyebrows of a suspect. How many different faces can be produced with this information?

Geometry **In Exercises 61–64, find the number of diagonals of the polygon. (A line segment connecting any two nonadjacent vertices is called a *diagonal* of the polygon.)**

61. Pentagon **62.** Hexagon

63. Octagon **64.** Decagon (10 sides)

Model It

65. ***Lottery*** Powerball is a lottery game that is operated by the Multi-State Lottery Association and is played in 27 states, Washington D.C., and the U.S. Virgin Islands. The game is played by drawing five white balls out of a drum of 53 white balls (numbered 1–53) and one red powerball out of a drum of 42 red balls (numbered 1–42). The jackpot is won by matching all five white balls in any order and the red powerball.

(a) Find the possible number of winning Powerball numbers.

(b) Find the possible number of winning Powerball numbers if the jackpot is won by matching all five white balls in order and the red power ball.

(c) Compare the results of part (a) with a state lottery in which a jackpot is won by matching six balls from a drum of 53 balls.

66. ***Permutations or Combinations?*** Decide whether each scenario should be counted using permutations or combinations. Explain your reasoning.

(a) Number of ways 10 people can line up in a row for concert tickets

(b) Number of different arrangements of three types of flowers from an array of 20 types

(c) Number of three-digit pin numbers for a debit card

(d) Number of two-scoop ice cream cones created from 31 different flavors

Synthesis

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. The number of letter pairs that can be formed in any order from any of the first 13 letters in the alphabet (A–M) is an example of a permutation.

68. The number of permutations of n elements can be determined by using the Fundamental Counting Principle.

69. What is the relationship between ${}_nC_r$ and ${}_nC_{n-r}$?

70. Without calculating the numbers, determine which of the following is greater. Explain.

(a) The number of combinations of 10 elements taken six at a time

(b) The number of permutations of 10 elements taken six at a time

Proof **In Exercises 71–74, prove the identity.**

71. ${}_nP_{n-1} = {}_nP_n$ **72.** ${}_nC_n = {}_nC_0$

73. ${}_nC_{n-1} = {}_nC_1$ **74.** ${}_nC_r = \dfrac{{}_nP_r}{r!}$

75. ***Think About It*** Can your calculator evaluate ${}_{100}P_{80}$? If not, explain why.

76. ***Writing*** Explain in words the meaning of ${}_nP_r$.

Skills Review

In Exercises 77–80, evaluate the function at each specified value of the independent variable and simplify.

77. $f(x) = 3x^2 + 8$

(a) $f(3)$ (b) $f(0)$ (c) $f(-5)$

78. $g(x) = \sqrt{x-3} + 2$

(a) $g(3)$ (b) $g(7)$ (c) $g(x+1)$

79. $f(x) = -|x-5| + 6$

(a) $f(-5)$ (b) $f(-1)$ (c) $f(11)$

80. $f(x) = \begin{cases} x^2 - 2x + 5, & x \le -4 \\ -x^2 - 2, & x > -4 \end{cases}$

(a) $f(-4)$ (b) $f(-1)$ (c) $f(-20)$

In Exercises 81–84, solve the equation. Round your answer to two decimal places, if necessary.

81. $\sqrt{x-3} = x - 6$ **82.** $\dfrac{4}{t} + \dfrac{3}{2t} = 1$

83. $\log_2(x-3) = 5$ **84.** $e^{x/3} = 16$