9.3 Geometric Sequences and Series

What you should learn

- Recognize, write, and find the *n*th terms of geometric sequences.
- Find *n*th partial sums of geometric sequences.
- Find the sum of an infinite geometric series.
- Use geometric sequences to model and solve real-life problems.

Why you should learn it

Geometric sequences can be used to model and solve real-life problems. For instance, in Exercise 99 on page 670, you will use a geometric sequence to model the population of China.

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Geometric Sequences

In Section 9.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence.** Consecutive terms of a geometric sequence have a common *ratio*.

Definition of Geometric Sequence

A sequence is **geometric** if the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is geometric if there is a number r such that

$$\frac{a_2}{a_1} = r, \quad \frac{a_3}{a_2} = r, \quad \frac{a_4}{a_3} = r, \quad r \neq 0$$

and so the number r is the **common ratio** of the sequence.

Example 1 Examples of Geometric Sequences

a. The sequence whose nth term is 2^n is geometric. For this sequence, the common ratio of consecutive terms is 2.

$$\underbrace{2, 4,}_{\frac{4}{2} = 2}\ 8, 16, \ldots, 2^n, \ldots \qquad \text{Begin with } n = 1.$$

b. The sequence whose nth term is $4(3^n)$ is geometric. For this sequence, the common ratio of consecutive terms is 3.

$$\underbrace{12, 36,}_{\frac{36}{12} = 3}\ 108, 324, \ldots, 4(3^n), \ldots \qquad \text{Begin with } n = 1.$$

c. The sequence whose nth term is $\left(-\frac{1}{3}\right)^n$ is geometric. For this sequence, the common ratio of consecutive terms is $-\frac{1}{3}$.

$$\underbrace{-\frac{1}{3}, \frac{1}{9},}_{\frac{1/9}{-1/3} = -\frac{1}{3}}\ -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots \qquad \text{Begin with } n = 1.$$

CHECKPOINT Now try Exercise 1.

The sequence 1, 4, 9, 16, . . . , whose nth term is n^2, is *not* geometric. The ratio of the second term to the first term is

$$\frac{a_2}{a_1} = \frac{4}{1} = 4$$

but the ratio of the third term to the second term is $\frac{a_3}{a_2} = \frac{9}{4}$.

In Example 1, notice that each of the geometric sequences has an nth term that is of the form ar^n, where the common ratio of the sequence is r. A geometric sequence may be thought of as an exponential function whose domain is the set of natural numbers.

The nth Term of a Geometric Sequence

The nth term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$\begin{array}{cccccccc} a_1, & a_2, & a_3, & a_4, & a_5, & \ldots, & a_n, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ a_1, & a_1 r, & a_1 r^2, & a_1 r^3, & a_1 r^4, & \ldots, & a_1 r^{n-1}, & \ldots \end{array}$$

Additional Example

Write the first five terms of the geometric sequence whose first term is $a_1 = 9$ and whose common ratio is $r = \frac{1}{3}$.

Solution

$9, 3, 1, \frac{1}{3}, \frac{1}{9}$

If you know the nth term of a geometric sequence, you can find the $(n + 1)$th term by multiplying by r. That is, $a_{n+1} = ra_n$.

Example 2 Finding the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$. Then graph the terms on a set of coordinate axes.

Solution

Starting with 3, repeatedly multiply by 2 to obtain the following.

$a_1 = 3$	1st term
$a_2 = 3(2^1) = 6$	2nd term
$a_3 = 3(2^2) = 12$	3rd term
$a_4 = 3(2^3) = 24$	4th term
$a_5 = 3(2^4) = 48$	5th term

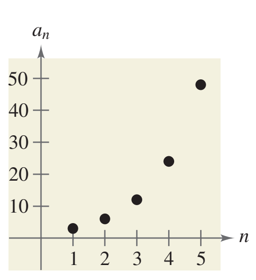


FIGURE 9.5

Figure 9.5 shows the first five terms of this geometric sequence.

CHECKPOINT Now try Exercise 11.

Example 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

Solution

$a_{15} = a_1 r^{n-1}$	Formula for geometric sequence
$= 20(1.05)^{15-1}$	Substitute 20 for a_1, 1.05 for r, and 15 for n.
≈ 39.599	Use a calculator.

CHECKPOINT Now try Exercise 27.

Example 4 Finding a Term of a Geometric Sequence

Find the 12th term of the geometric sequence

$5, 15, 45, \ldots.$

Solution

The common ratio of this sequence is

$$r = \frac{15}{5} = 3.$$

Because the first term is $a_1 = 5$, you can determine the 12th term $(n = 12)$ to be

$$a_n = a_1 r^{n-1} \qquad \text{Formula for geometric sequence}$$

$$a_{12} = 5(3)^{12-1} \qquad \text{Substitute 5 for } a_1\text{, 3 for } r\text{, and 12 for } n.$$

$$= 5(177{,}147) \qquad \text{Use a calculator.}$$

$$= 885{,}735. \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 35.

If you know any two terms of a geometric sequence, you can use that information to find a formula for the nth term of the sequence.

STUDY TIP

Remember that r is the common ratio of consecutive terms of a geometric sequence. So, in Example 5,

$$\begin{aligned} a_{10} &= a_1 r^9 \\ &= a_1 \cdot r \cdot r \cdot r \cdot r^6 \\ &= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6 \\ &= a_4 r^6. \end{aligned}$$

Example 5 Finding a Term of a Geometric Sequence

The fourth term of a geometric sequence is 125, and the 10th term is 125/64. Find the 14th term. (Assume that the terms of the sequence are positive.)

Solution

The 10th term is related to the fourth term by the equation

$$a_{10} = a_4 r^6. \qquad \text{Multiply 4th term by } r^{10-4}.$$

Because $a_{10} = 125/64$ and $a_4 = 125$, you can solve for r as follows.

$$\frac{125}{64} = 125r^6 \qquad \text{Substitute } \tfrac{125}{64} \text{ for } a_{10} \text{ and 125 for } a_4.$$

$$\frac{1}{64} = r^6 \qquad \text{Divide each side by 125.}$$

$$\frac{1}{2} = r \qquad \text{Take the sixth root of each side.}$$

You can obtain the 14th term by multiplying the 10th term by r^4.

$$a_{14} = a_{10} r^4 \qquad \text{Multiply the 10th term by } r^{14-10}.$$

$$= \frac{125}{64}\left(\frac{1}{2}\right)^4 \qquad \text{Substitute } \tfrac{125}{64} \text{ for } a_{10} \text{ and } \tfrac{1}{2} \text{ for } r.$$

$$= \frac{125}{1024} \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 41.

The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

The Sum of a Finite Geometric Sequence

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ .\ .\ .\ ,\ a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

For a proof of the sum of a finite geometric sequence, see Proofs in Mathematics on page 723.

Example 6 Finding the Sum of a Finite Geometric Sequence

Find the sum $\sum_{i=1}^{12} 4(0.3)^{i-1}$.

Solution

By writing out a few terms, you have

$$\sum_{i=1}^{12} 4(0.3)^{i-1} = 4(0.3)^0 + 4(0.3)^1 + 4(0.3)^2 + \cdot \cdot \cdot + 4(0.3)^{11}.$$

Now, because $a_1 = 4$, $r = 0.3$, and $n = 12$, you can apply the formula for the sum of a finite geometric sequence to obtain

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right) \qquad \text{Formula for the sum of a sequence}$$

$$\sum_{i=1}^{12} 4(0.3)^{i-1} = 4\left[\frac{1-(0.3)^{12}}{1-0.3}\right] \qquad \text{Substitute 4 for } a_1,\ 0.3 \text{ for } r, \text{ and 12 for } n.$$

$$\approx 5.714. \qquad \text{Use a calculator.}$$

CHECKPOINT Now try Exercise 57.

When using the formula for the sum of a finite geometric sequence, be careful to check that the sum is of the form

$$\sum_{i=1}^{n} a_1 r^{i-1}. \qquad \text{Exponent for } r \text{ is } i-1.$$

If the sum is not of this form, you must adjust the formula. For instance, if the sum in Example 6 were $\sum_{i=1}^{12} 4(0.3)^i$, then you would evaluate the sum as follows.

$$\sum_{i=1}^{12} 4(0.3)^i = 4(0.3) + 4(0.3)^2 + 4(0.3)^3 + \cdot \cdot \cdot + 4(0.3)^{12}$$

$$= 4(0.3) + [4(0.3)](0.3) + [4(0.3)](0.3)^2 + \cdot \cdot \cdot + [4(0.3)](0.3)^{11}$$

$$= 4(0.3)\left[\frac{1-(0.3)^{12}}{1-0.3}\right] \approx 1.714. \qquad a_1 = 4(0.3),\ r = 0.3,\ n = 12$$

Exploration

Use a graphing utility to graph

$$y = \left(\frac{1 - r^x}{1 - r}\right)$$

for $r = \frac{1}{2}, \frac{2}{3}$, and $\frac{4}{5}$. What happens as $x \to \infty$?

Use a graphing utility to graph

$$y = \left(\frac{1 - r^x}{1 - r}\right)$$

for $r = 1.5$, 2, and 3. What happens as $x \to \infty$?

Geometric Series

The summation of the terms of an infinite geometric *sequence* is called an **infinite geometric series** or simply a **geometric series.**

The formula for the sum of a *finite* geometric *sequence* can, depending on the value of r, be extended to produce a formula for the sum of an *infinite* geometric *series*. Specifically, if the common ratio r has the property that $|r| < 1$, it can be shown that r^n becomes arbitrarily close to zero as n increases without bound. Consequently,

$$a_1\left(\frac{1 - r^n}{1 - r}\right) \longrightarrow a_1\left(\frac{1 - 0}{1 - r}\right) \quad \text{as} \quad n \longrightarrow \infty.$$

This result is summarized as follows.

The Sum of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + \cdots$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1 - r}.$$

Note that if $|r| \geq 1$, the series does not have a sum.

Infinite geometric series and their sums have important uses in calculus. Refer to Exercise 120 in this section.

Activities

1. Determine which of the following are geometric sequences.
   (a) $3, 6, 9, 12, 15, \ldots$
   (b) $2, 4, 8, 16, 32, \ldots$
   (c) $1, -1, 1, -1, 1, \ldots$
   (d) $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$
   (e) $2, 4, 16, 64, 256, \ldots$
   Answer: (b), (c), (d)
2. Find the sum.
   $$\sum_{n=1}^{10} 16\left(\tfrac{1}{2}\right)^n$$
   Answer: ≈ 15.984
3. Find the sum.
   $$\sum_{n=0}^{\infty} 16\left(\tfrac{1}{2}\right)^n$$
   Answer: 32

Example 7 Finding the Sum of an Infinite Geometric Series

Find each sum.

a. $\displaystyle\sum_{n=1}^{\infty} 4(0.6)^{n-1}$

b. $3 + 0.3 + 0.03 + 0.003 + \cdots$

Solution

a.
$$\begin{aligned}\sum_{n=1}^{\infty} 4(0.6)^{n-1} &= 4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots + 4(0.6)^{n-1} + \cdots \\ &= \frac{4}{1 - 0.6} \qquad \frac{a_1}{1 - r} \\ &= 10\end{aligned}$$

b.
$$\begin{aligned}3 + 0.3 + 0.03 + 0.003 + \cdots &= 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots \\ &= \frac{3}{1 - 0.1} \qquad \frac{a_1}{1 - r} \\ &= \frac{10}{3} \\ &\approx 3.33\end{aligned}$$

CHECKPOINT Now try Exercise 79.

Application

Example 8 Increasing Annuity

A deposit of \$50 is made on the first day of each month in a savings account that pays 6% compounded monthly. What is the balance at the end of 2 years? (This type of savings plan is called an **increasing annuity.**)

Solution

The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1 + \frac{0.06}{12}\right)^{24}$$
$$= 50(1.005)^{24}.$$

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1 + \frac{0.06}{12}\right)^{23}$$
$$= 50(1.005)^{23}.$$

The last deposit will gain interest for only 1 month, and its balance will be

$$A_1 = 50\left(1 + \frac{0.06}{12}\right)^{1}$$
$$= 50(1.005).$$

The total balance in the annuity will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.005)$ and $r = 1.005$, you have

$$S_{24} = 50(1.005)\left[\frac{1 - (1.005)^{24}}{1 - 1.005}\right]$$ Substitute $50(1.005)$ for A_1, 1.005 for r, and 24 for n.

$$= \$1277.96.$$ Simplify.

CHECKPOINT Now try Exercise 107.

STUDY TIP

Recall from Section 3.1 that the formula for compound interest is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

So, in Example 8, \$50 is the principal P, 0.06 is the interest rate r, 12 is the number of compoundings per year n, and 2 is the time t in years. If you substitute these values into the formula, you obtain

$$A = 50\left(1 + \frac{0.06}{12}\right)^{12(2)}$$
$$= 50\left(1 + \frac{0.06}{12}\right)^{24}.$$

Writing About Mathematics

An Experiment You will need a piece of string or yarn, a pair of scissors, and a tape measure. Measure out any length of string at least 5 feet long. Double over the string and cut it in half. Take one of the resulting halves, double it over, and cut it in half. Continue this process until you are no longer able to cut a length of string in half. How many cuts were you able to make? Construct a sequence of the resulting string lengths after each cut, starting with the original length of the string. Find a formula for the nth term of this sequence. How many cuts could you theoretically make? Discuss why you were not able to make that many cuts.

Writing About Mathematics Suggestion: For the sake of simplicity, you may want to consider supplying each group with the tools they need for this activity rather than having them supply the materials themselves.

9.3 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. A sequence is called a ________ sequence if the ratios between consecutive terms are the same. This ratio is called the ________ ratio.
2. The nth term of a geometric sequence has the form ________.
3. The formula for the sum of a finite geometric sequence is given by ________.
4. The sum of the terms of an infinite geometric sequence is called a ________ ________.
5. The formula for the sum of an infinite geometric series is given by ________.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–10, determine whether the sequence is geometric. If so, find the common ratio.

1. $5, 15, 45, 135, \ldots$

2. $3, 12, 48, 192, \ldots$

3. $3, 12, 21, 30, \ldots$

4. $36, 27, 18, 9, \ldots$

5. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

6. $5, 1, 0.2, 0.04, \ldots$

7. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \ldots$

8. $9, -6, 4, -\frac{8}{3}, \ldots$

9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

10. $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \ldots$

In Exercises 11–20, write the first five terms of the geometric sequence.

11. $a_1 = 2, r = 3$

12. $a_1 = 6, r = 2$

13. $a_1 = 1, r = \frac{1}{2}$

14. $a_1 = 1, r = \frac{1}{3}$

15. $a_1 = 5, r = -\frac{1}{10}$

16. $a_1 = 6, r = -\frac{1}{4}$

17. $a_1 = 1, r = e$

18. $a_1 = 3, r = \sqrt{5}$

19. $a_1 = 2, r = \dfrac{x}{4}$

20. $a_1 = 5, r = 2x$

In Exercises 21–26, write the first five terms of the geometric sequence. Determine the common ratio and write the nth term of the sequence as a function of n.

21. $a_1 = 64, \quad a_{k+1} = \frac{1}{2}a_k$

22. $a_1 = 81, \quad a_{k+1} = \frac{1}{3}a_k$

23. $a_1 = 7, \quad a_{k+1} = 2a_k$

24. $a_1 = 5, \quad a_{k+1} = -2a_k$

25. $a_1 = 6, \quad a_{k+1} = -\frac{3}{2}a_k$

26. $a_1 = 48, \quad a_{k+1} = -\frac{1}{2}a_k$

In Exercises 27–34, write an expression for the nth term of the geometric sequence. Then find the indicated term.

27. $a_1 = 4, r = \frac{1}{2}, n = 10$

28. $a_1 = 5, r = \frac{3}{2}, n = 8$

29. $a_1 = 6, r = -\frac{1}{3}, n = 12$

30. $a_1 = 64, r = -\frac{1}{4}, n = 10$

31. $a_1 = 100, r = e^x, n = 9$

32. $a_1 = 1, r = \sqrt{3}, n = 8$

33. $a_1 = 500, r = 1.02, n = 40$

34. $a_1 = 1000, r = 1.005, n = 60$

In Exercises 35–42, find the indicated nth term of the geometric sequence.

35. 9th term: $7, 21, 63, \ldots$

36. 7th term: $3, 36, 432, \ldots$

37. 10th term: $5, 30, 180, \ldots$

38. 22nd term: $4, 8, 16, \ldots$

39. 3rd term: $a_1 = 16, a_4 = \frac{27}{4}$

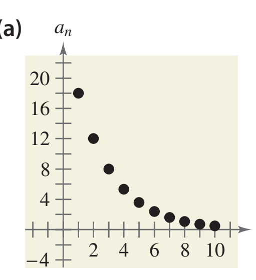
40. 1st term: $a_2 = 3, a_5 = \frac{3}{64}$

41. 6th term: $a_4 = -18, a_7 = \frac{2}{3}$

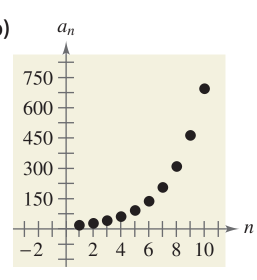
42. 7th term: $a_3 = \frac{16}{3}, a_5 = \frac{64}{27}$

In Exercises 43–46, match the geometric sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

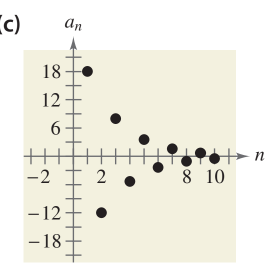
(a)



(b)



(c)



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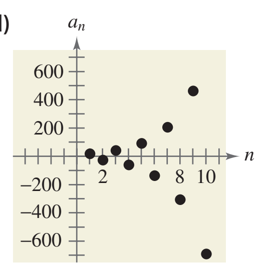


43. $a_n = 18\left(\frac{2}{3}\right)^{n-1}$

44. $a_n = 18\left(-\frac{2}{3}\right)^{n-1}$

45. $a_n = 18\left(\frac{3}{2}\right)^{n-1}$

46. $a_n = 18\left(-\frac{3}{2}\right)^{n-1}$

In Exercises 47–52, use a graphing utility to graph the first 10 terms of the sequence.

47. $a_n = 12(-0.75)^{n-1}$ **48.** $a_n = 10(1.5)^{n-1}$

49. $a_n = 12(-0.4)^{n-1}$ **50.** $a_n = 20(-1.25)^{n-1}$

51. $a_n = 2(1.3)^{n-1}$ **52.** $a_n = 10(1.2)^{n-1}$

In Exercises 53–72, find the sum of the finite geometric sequence.

53. $\sum_{n=1}^{9} 2^{n-1}$ **54.** $\sum_{n=1}^{10} \left(\frac{5}{2}\right)^{n-1}$

55. $\sum_{n=1}^{9} (-2)^{n-1}$ **56.** $\sum_{n=1}^{8} 5\left(-\frac{3}{2}\right)^{n-1}$

57. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1}$ **58.** $\sum_{i=1}^{10} 2\left(\frac{1}{4}\right)^{i-1}$

59. $\sum_{i=1}^{6} 32\left(\frac{1}{4}\right)^{i-1}$ **60.** $\sum_{i=1}^{12} 16\left(\frac{1}{2}\right)^{i-1}$

61. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^{n}$ **62.** $\sum_{n=0}^{40} 5\left(\frac{3}{5}\right)^{n}$

63. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^{n}$ **64.** $\sum_{n=0}^{20} 10\left(\frac{1}{5}\right)^{n}$

65. $\sum_{n=0}^{5} 300(1.06)^{n}$ **66.** $\sum_{n=0}^{6} 500(1.04)^{n}$

67. $\sum_{n=0}^{40} 2\left(-\frac{1}{4}\right)^{n}$ **68.** $\sum_{n=0}^{50} 10\left(\frac{2}{3}\right)^{n-1}$

69. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1}$ **70.** $\sum_{i=0}^{25} 8\left(-\frac{1}{2}\right)^{i}$

71. $\sum_{i=1}^{10} 5\left(-\frac{1}{3}\right)^{i-1}$ **72.** $\sum_{i=1}^{100} 15\left(\frac{2}{3}\right)^{i-1}$

In Exercises 73–78, use summation notation to write the sum.

73. $5 + 15 + 45 + \cdots + 3645$

74. $7 + 14 + 28 + \cdots + 896$

75. $2 - \frac{1}{2} + \frac{1}{8} - \cdots + \frac{1}{2048}$

76. $15 - 3 + \frac{3}{5} - \cdots - \frac{3}{625}$

77. $0.1 + 0.4 + 1.6 + \cdots + 102.4$

78. $32 + 24 + 18 + \cdots + 10.125$

In Exercises 79–92, find the sum of the infinite geometric series.

79. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n}$ **80.** $\sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^{n}$

81. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n}$ **82.** $\sum_{n=0}^{\infty} 2\left(-\frac{2}{3}\right)^{n}$

83. $\sum_{n=0}^{\infty} 4\left(\frac{1}{4}\right)^{n}$ **84.** $\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^{n}$

85. $\sum_{n=0}^{\infty} (0.4)^{n}$ **86.** $\sum_{n=0}^{\infty} 4(0.2)^{n}$

87. $\sum_{n=0}^{\infty} -3(0.9)^{n}$ **88.** $\sum_{n=0}^{\infty} -10(0.2)^{n}$

89. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$ **90.** $9 + 6 + 4 + \frac{8}{3} + \cdots$

91. $\frac{1}{9} - \frac{1}{3} + 1 - 3 + \cdots$ **92.** $-\frac{125}{36} + \frac{25}{6} - 5 + 6 - \cdots$

In Exercises 93–96, find the rational number representation of the repeating decimal.

93. $0.\overline{36}$ **94.** $0.\overline{297}$

95. $0.3\overline{18}$ **96.** $1.3\overline{8}$

Graphical Reasoning **In Exercises 97 and 98, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum.**

97. $f(x) = 6\left[\frac{1 - (0.5)^x}{1 - (0.5)}\right], \quad \sum_{n=0}^{\infty} 6\left(\frac{1}{2}\right)^n$

98. $f(x) = 2\left[\frac{1 - (0.8)^x}{1 - (0.8)}\right], \quad \sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$

Model It

99. ***Data Analysis: Population*** The table shows the population a_n of China (in millions) from 1998 through 2004. (Source: U.S. Census Bureau)

Year	Population, a_n
1998	1250.4
1999	1260.1
2000	1268.9
2001	1276.9
2002	1284.3
2003	1291.5
2004	1298.8

(a) Use the *exponential regression* feature of a graphing utility to find a geometric sequence that models the data. Let n represent the year, with $n = 8$ corresponding to 1998.

(b) Use the sequence from part (a) to describe the rate at which the population of China is growing.

Model It (continued)

(c) Use the sequence from part (a) to predict the population of China in 2010. The U.S. Census Bureau predicts the population of China will be 1374.6 million in 2010. How does this value compare with your prediction?

(d) Use the sequence from part (a) to determine when the population of China will reach 1.32 billion.

100. ***Compound Interest*** A principal of \$1000 is invested at 6% interest. Find the amount after 10 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

101. ***Compound Interest*** A principal of \$2500 is invested at 2% interest. Find the amount after 20 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

102. ***Depreciation*** A tool and die company buys a machine for \$135,000 and it depreciates at a rate of 30% per year. (In other words, at the end of each year the depreciated value is 70% of what it was at the beginning of the year.) Find the depreciated value of the machine after 5 full years.

103. ***Annuities*** A deposit of \$100 is made at the beginning of each month in an account that pays 6%, compounded monthly. The balance A in the account at the end of 5 years is

$$A = 100\left(1 + \frac{0.06}{12}\right)^1 + \cdots + 100\left(1 + \frac{0.06}{12}\right)^{60}.$$

Find A.

104. ***Annuities*** A deposit of \$50 is made at the beginning of each month in an account that pays 8%, compounded monthly. The balance A in the account at the end of 5 years is

$$A = 50\left(1 + \frac{0.08}{12}\right)^1 + \cdots + 50\left(1 + \frac{0.08}{12}\right)^{60}.$$

Find A.

105. ***Annuities*** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded monthly. The balance A after t years is

$$A = P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^2 + \cdots + P\left(1 + \frac{r}{12}\right)^{12t}.$$

Show that the balance is

$$A = P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right).$$

106. ***Annuities*** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded continuously. The balance A after t years is $A = Pe^{r/12} + Pe^{2r/12} + \cdots + Pe^{12tr/12}$. Show that the balance is

$$A = \frac{Pe^{r/12}(e^{rt} - 1)}{e^{r/12} - 1}.$$

Annuities **In Exercises 107–110, consider making monthly deposits of P dollars in a savings account earning an annual interest rate r. Use the results of Exercises 105 and 106 to find the balance A after t years if the interest is compounded (a) monthly and (b) continuously.**

107. $P = \$50,\ r = 7\%,\ t = 20$ years

108. $P = \$75,\ r = 3\%,\ t = 25$ years

109. $P = \$100,\ r = 10\%,\ t = 40$ years

110. $P = \$20,\ r = 6\%,\ t = 50$ years

111. ***Annuities*** Consider an initial deposit of P dollars in an account earning an annual interest rate r, compounded monthly. At the end of each month, a withdrawal of W dollars will occur and the account will be depleted in t years. The amount of the initial deposit required is

$$P = W\left(1 + \frac{r}{12}\right)^{-1} + W\left(1 + \frac{r}{12}\right)^{-2} + \cdots + W\left(1 + \frac{r}{12}\right)^{-12t}.$$

Show that the initial deposit is

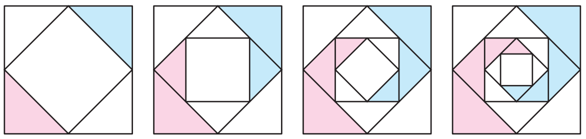
$$P = W\left(\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^{-12t}\right].$$

112. ***Annuities*** Determine the amount required in a retirement account for an individual who retires at age 65 and wants an income of \$2000 from the account each month for 20 years. Use the result of Exercise 111 and assume that the account earns 9% compounded monthly.

Multiplier Effect **In Exercises 113–116, use the following information. A tax rebate has been given to property owners by the state government with the anticipation that each property owner spends approximately p% of the rebate, and in turn each recipient of this amount spends p% of what they receive, and so on. Economists refer to this exchange of money and its circulation within the economy as the "multiplier effect." The multiplier effect operates on the idea that the expenditures of one individual become the income of another individual. For the given tax rebate, find the total amount put back into the state's economy, if this effect continues without end.**

	Tax rebate	$p\%$
113.	\$400	75%
114.	\$250	80%
115.	\$600	72.5%
116.	\$450	77.5%

117. ***Geometry*** The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the resulting triangles are shaded (see figure). If this process is repeated five more times, determine the total area of the shaded region.



118. ***Sales*** The annual sales a_n (in millions of dollars) for Urban Outfitters for 1994 through 2003 can be approximated by the model

$$a_n = 54.6e^{0.172n}, \qquad n = 4, 5, . . . , 13$$

where n represents the year, with $n = 4$ corresponding to 1994. Use this model and the formula for the sum of a finite geometric sequence to approximate the total sales earned during this 10-year period. (Source: Urban Outfitters Inc.)

119. ***Salary*** An investment firm has a job opening with a salary of \$30,000 for the first year. Suppose that during the next 39 years, there is a 5% raise each year. Find the total compensation over the 40-year period.

120. ***Distance*** A ball is dropped from a height of 16 feet. Each time it drops h feet, it rebounds $0.81h$ feet.

(a) Find the total vertical distance traveled by the ball.

(b) The ball takes the following times (in seconds) for each fall.

$$\begin{array}{ll} s_1 = -16t^2 + 16, & s_1 = 0 \text{ if } t = 1 \\ s_2 = -16t^2 + 16(0.81), & s_2 = 0 \text{ if } t = 0.9 \\ s_3 = -16t^2 + 16(0.81)^2, & s_3 = 0 \text{ if } t = (0.9)^2 \\ s_4 = -16t^2 + 16(0.81)^3, & s_4 = 0 \text{ if } t = (0.9)^3 \\ \vdots & \vdots \\ s_n = -16t^2 + 16(0.81)^{n-1}, & s_n = 0 \text{ if } t = (0.9)^{n-1} \end{array}$$

Beginning with s_2, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is

$$t = 1 + 2\sum_{n=1}^{\infty} (0.9)^n.$$

Find this total time.

Synthesis

True or False? **In Exercises 121 and 122, determine whether the statement is true or false. Justify your answer.**

121. A sequence is geometric if the ratios of consecutive differences of consecutive terms are the same.

122. You can find the nth term of a geometric sequence by multiplying its common ratio by the first term of the sequence raised to the $(n - 1)$th power.

123. ***Writing*** Write a brief paragraph explaining why the terms of a geometric sequence decrease in magnitude when $-1 < r < 1$.

124. Find two different geometric series with sums of 4.

Skills Review

In Exercises 125–128, evaluate the function for $f(x) = 3x + 1$ and $g(x) = x^2 - 1$.

125. $g(x + 1)$

126. $f(x + 1)$

127. $f(g(x + 1))$

128. $g(f(x + 1))$

In Exercises 129–132, completely factor the expression.

129. $9x^3 - 64x$

130. $x^2 + 4x - 63$

131. $6x^2 - 13x - 5$

132. $16x^2 - 4x^4$

In Exercises 133–138, perform the indicated operation(s) and simplify.

133. $\dfrac{3}{x + 3} \cdot \dfrac{x(x + 3)}{x - 3}$

134. $\dfrac{x - 2}{x + 7} \cdot \dfrac{2x(x + 7)}{6x(x - 2)}$

135. $\dfrac{x}{3} \div \dfrac{3x}{6x + 3}$

136. $\dfrac{x - 5}{x - 3} \div \dfrac{10 - 2x}{2(3 - x)}$

137. $5 + \dfrac{7}{x + 2} + \dfrac{2}{x - 2}$

138. $8 - \dfrac{x - 1}{x + 4} - \dfrac{4}{x - 1} - \dfrac{x + 4}{(x - 1)(x + 4)}$

139. **Make a Decision** To work an extended application analyzing the amounts spent on research and development in the United States from 1980 to 2003, visit this text's website at *college.hmco.com*. *(Data Source: U.S. Census Bureau)*