9.1 Sequences and Series

What you should learn

- Use sequence notation to write the terms of sequences.
- Use factorial notation.
- Use summation notation to write sums.
- Find the sums of infinite series.
- Use sequences and series to model and solve real-life problems.

Why you should learn it

Sequences and series can be used to model real-life problems. For instance, in Exercise 109 on page 651, sequences are used to model the number of Best Buy stores from 1998 through 2003.

Scott Olson/Getty Images

The *HM mathSpace*® CD-ROM and *Eduspace*® for this text contain additional resources related to the concepts discussed in this chapter.

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English. Saying that a collection is listed in *sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers.

$$f(1) = a_1,\ f(2) = a_2,\ f(3) = a_3,\ f(4) = a_4, \ldots, f(n) = a_n, \ldots$$

Rather than using function notation, however, sequences are usually written using subscript notation, as indicated in the following definition.

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

are the **terms** of the sequence. If the domain of the function consists of the first n positive integers only, the sequence is a **finite sequence.**

On occasion it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become $a_0, a_1, a_2, a_3, \ldots$.

Example 1 Writing the Terms of a Sequence

Write the first four terms of the sequences given by

a. $a_n = 3n - 2$ **b.** $a_n = 3 + (-1)^n$.

Solution

a. The first four terms of the sequence given by $a_n = 3n - 2$ are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10.$ 4th term

b. The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$a_1 = 3 + (-1)^1 = 3 - 1 = 2$ 1st term

$a_2 = 3 + (-1)^2 = 3 + 1 = 4$ 2nd term

$a_3 = 3 + (-1)^3 = 3 - 1 = 2$ 3rd term

$a_4 = 3 + (-1)^4 = 3 + 1 = 4.$ 4th term

CHECKPOINT Now try Exercise 1.

Example 2 A Sequence Whose Terms Alternate in Sign

Write the first five terms of the sequence given by $a_n = \dfrac{(-1)^n}{2n-1}$.

Solution

The first five terms of the sequence are as follows.

$$a_1 = \frac{(-1)^1}{2(1)-1} = \frac{-1}{2-1} = -1 \qquad \text{1st term}$$

$$a_2 = \frac{(-1)^2}{2(2)-1} = \frac{1}{4-1} = \frac{1}{3} \qquad \text{2nd term}$$

$$a_3 = \frac{(-1)^3}{2(3)-1} = \frac{-1}{6-1} = -\frac{1}{5} \qquad \text{3rd term}$$

$$a_4 = \frac{(-1)^4}{2(4)-1} = \frac{1}{8-1} = \frac{1}{7} \qquad \text{4th term}$$

$$a_5 = \frac{(-1)^5}{2(5)-1} = \frac{-1}{10-1} = -\frac{1}{9} \qquad \text{5th term}$$

CHECKPOINT Now try Exercise 17.

Simply listing the first few terms is not sufficient to define a unique sequence—the nth term *must be given*. To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

Example 3 Finding the nth Term of a Sequence

Write an expression for the apparent nth term (a_n) of each sequence.

a. $1, 3, 5, 7, \ldots$ **b.** $2, -5, 10, -17, \ldots$

Solution

a. *n:* $1 \;\; 2 \;\; 3 \;\; 4 \;\; \ldots \;\; n$

Terms: $1 \;\; 3 \;\; 5 \;\; 7 \;\; \ldots \;\; a_n$

Apparent pattern: Each term is 1 less than twice n, which implies that

$$a_n = 2n - 1.$$

b. *n:* $1 \;\; 2 \;\; 3 \;\; 4 \;\; \ldots \;\; n$

Terms: $2 \;\; -5 \;\; 10 \;\; -17 \;\; \ldots \;\; a_n$

Apparent pattern: The terms have alternating signs with those in the even positions being negative. Each term is 1 more than the square of n, which implies that

$$a_n = (-1)^{n+1}(n^2 + 1)$$

CHECKPOINT Now try Exercise 37.

Exploration

Write out the first five terms of the sequence whose nth term is

$$a_n = \frac{(-1)^{n+1}}{2n-1}.$$

Are they the same as the first five terms of the sequence in Example 2? If not, how do they differ?

Additional Example

Write an expression for the apparent nth term a_n of the sequence

$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$

Solution:

n: $1 \;\; 2 \;\; 3 \;\; 4 \;\; \ldots \;\; n$

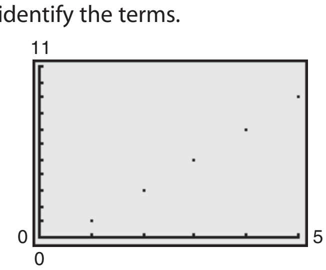
Terms: $\frac{2}{1} \;\; \frac{3}{2} \;\; \frac{4}{3} \;\; \frac{5}{4} \;\; \ldots \;\; a_n$

Apparent pattern: Each term has a numerator that is 1 greater than its denominator, which implies that

$a_n = \dfrac{n+1}{n}.$

Technology

To graph a sequence using a graphing utility, set the mode to *sequence* and *dot* and enter the sequence. The graph of the sequence in Example 3(a) is shown below. You can use the *trace* feature or *value* feature to identify the terms.



Some sequences are defined **recursively.** To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms. A well-known example is the Fibonacci sequence shown in Example 4.

Example 4 The Fibonacci Sequence: A Recursive Sequence

The Fibonacci sequence is defined recursively, as follows.

$$a_0 = 1, \; a_1 = 1, \; a_k = a_{k-2} + a_{k-1}, \text{ where } k \geq 2$$

Write the first six terms of this sequence.

Solution

$a_0 = 1$ — 0th term is given.

$a_1 = 1$ — 1st term is given.

$a_2 = a_{2-2} + a_{2-1} = a_0 + a_1 = 1 + 1 = 2$ — Use recursion formula.

$a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 1 + 2 = 3$ — Use recursion formula.

$a_4 = a_{4-2} + a_{4-1} = a_2 + a_3 = 2 + 3 = 5$ — Use recursion formula.

$a_5 = a_{5-2} + a_{5-1} = a_3 + a_4 = 3 + 5 = 8$ — Use recursion formula.

✓CHECKPOINT Now try Exercise 51.

STUDY TIP

The subscripts of a sequence make up the domain of the sequence and they serve to identify the location of a term within the sequence. For example, a_4 is the fourth term of the sequence, and a_n is the nth term of the sequence. Any variable can be used as a subscript. The most commonly used variable subscripts in sequence and series notation are i, j, k, and n.

Factorial Notation

Some very important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

Definition of Factorial

If n is a positive integer, ***n*** **factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n-1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

Here are some values of $n!$ for the first several nonnegative integers. Notice that $0!$ is 1 by definition.

$$0! = 1$$
$$1! = 1$$
$$2! = 1 \cdot 2 = 2$$
$$3! = 1 \cdot 2 \cdot 3 = 6$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$
$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

The value of n does not have to be very large before the value of $n!$ becomes extremely large. For instance, $10! = 3{,}628{,}800$.

Factorials follow the same conventions for order of operations as do exponents. For instance,

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot n)$$

whereas $(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot 2n$.

Example 5 Writing the Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by

$$a_n = \frac{2^n}{n!}.$$

Begin with $n = 0$. Then graph the terms on a set of coordinate axes.

Solution

$$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1 \qquad \text{0th term}$$

$$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2 \qquad \text{1st term}$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2 \qquad \text{2nd term}$$

$$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3} \qquad \text{3rd term}$$

$$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3} \qquad \text{4th term}$$

Figure 9.1 shows the first five terms of the sequence.

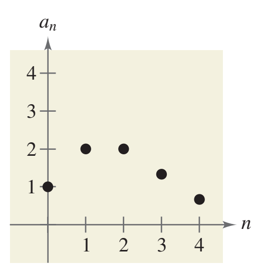


FIGURE 9.1

Additional Examples

a. $\dfrac{(2n + 2)!}{(2n + 4)!} = \dfrac{1}{(2n + 4)(2n + 3)}$

b. $\dfrac{2n!}{n!} = 2$

c. $\dfrac{(2n + 1)!}{(2n)!} = 2n + 1$

You may want to point out to your students that $n! = n \cdot (n - 1)!$

CHECKPOINT Now try Exercise 59.

When working with fractions involving factorials, you will often find that the fractions can be reduced to simplify the computations.

Example 6 Evaluating Factorial Expressions

Evaluate each factorial expression.

a. $\dfrac{8!}{2! \cdot 6!}$ **b.** $\dfrac{2! \cdot 6!}{3! \cdot 5!}$ **c.** $\dfrac{n!}{(n - 1)!}$

Solution

a. $\dfrac{8!}{2! \cdot 6!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 7 \cdot 8}{1 \cdot 2 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}} = \dfrac{7 \cdot 8}{2} = 28$

b. $\dfrac{2! \cdot 6!}{3! \cdot 5!} = \dfrac{\cancel{1 \cdot 2} \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot 6}{\cancel{1 \cdot 2} \cdot 3 \cdot \cancel{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}} = \dfrac{6}{3} = 2$

c. $\dfrac{n!}{(n - 1)!} = \dfrac{\cancel{1 \cdot 2 \cdot 3 \cdot \cdot \cdot (n - 1)} \cdot n}{\cancel{1 \cdot 2 \cdot 3 \cdot \cdot \cdot (n - 1)}} = n$

CHECKPOINT Now try Exercise 69.

STUDY TIP

Note in Example 6(a) that you can simplify the computation as follows.

$$\frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7 \cdot \cancel{6!}}{2! \cdot \cancel{6!}}$$

$$= \frac{8 \cdot 7}{2 \cdot 1} = 28$$

Technology

Most graphing utilities are able to sum the first n terms of a sequence. Check your user's guide for a *sum sequence* feature or a *series* feature.

Reading and writing the upper and lower limits of summation correctly will help with problems involving upper and lower limits in calculus.

STUDY TIP

Summation notation is an instruction to add the terms of a sequence. From the definition at the right, the upper limit of summation tells you where to end the sum. Summation notation helps you generate the appropriate terms of the sequence prior to finding the actual sum, which may be unclear.

Summation Notation

There is a convenient notation for the sum of the terms of a finite sequence. It is called **summation notation** or **sigma notation** because it involves the use of the uppercase Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where i is called the **index of summation,** n is the **upper limit of summation,** and 1 is the **lower limit of summation.**

Example 7 Summation Notation for Sums

Find each sum.

a. $\displaystyle\sum_{i=1}^{5} 3i$ **b.** $\displaystyle\sum_{k=3}^{6} (1 + k^2)$ **c.** $\displaystyle\sum_{i=0}^{8} \frac{1}{i!}$

Solution

a. $$\sum_{i=1}^{5} 3i = 3(1) + 3(2) + 3(3) + 3(4) + 3(5)$$
$$= 3(1 + 2 + 3 + 4 + 5)$$
$$= 3(15)$$
$$= 45$$

b. $$\sum_{k=3}^{6} (1 + k^2) = (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2)$$
$$= 10 + 17 + 26 + 37$$
$$= 90$$

c. $$\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$$
$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}$$
$$\approx 2.71828$$

For this summation, note that the sum is very close to the irrational number $e \approx 2.718281828$. It can be shown that as more terms of the sequence whose nth term is $1/n!$ are added, the sum becomes closer and closer to e.

CHECKPOINT Now try Exercise 73.

In Example 7, note that the lower limit of a summation does not have to be 1. Also note that the index of summation does not have to be the letter i. For instance, in part (b), the letter k is the index of summation.

STUDY TIP

Variations in the upper and lower limits of summation can produce quite different-looking summation notations for *the same sum*. For example, the following two sums have the same terms.

$$\sum_{i=1}^{3} 3(2^i) = 3(2^1 + 2^2 + 2^3)$$

$$\sum_{i=0}^{2} 3(2^{i+1}) = 3(2^1 + 2^2 + 2^3)$$

Properties of Sums

1. $\sum_{i=1}^{n} c = cn$, c is a constant.
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, c is a constant.
3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

For proofs of these properties, see Proofs in Mathematics on page 722.

Series

Many applications involve the sum of the terms of a finite or infinite sequence. Such a sum is called a **series.**

Definition of Series

Consider the infinite sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$.

1. The sum of the first n terms of the sequence is called a **finite series** or the ***n*th partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

2. The sum of all the terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

Example 8 Finding the Sum of a Series

For the series $\sum_{i=1}^{\infty} \frac{3}{10^i}$, find (a) the third partial sum and (b) the sum.

Solution

a. The third partial sum is

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} = 0.3 + 0.03 + 0.003 = 0.333.$$

b. The sum of the series is

$$\sum_{i=1}^{\infty} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots$$
$$= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots$$
$$= 0.33333\ldots = \frac{1}{3}.$$

CHECKPOINT Now try Exercise 99.

Application

Sequences have many applications in business and science. One such application is illustrated in Example 9.

Example 9 Population of the United States

For the years 1980 to 2003, the resident population of the United States can be approximated by the model

$$a_n = 226.9 + 2.05n + 0.035n^2, \qquad n = 0, 1, \ldots, 23$$

where a_n is the population (in millions) and n represents the year, with $n = 0$ corresponding to 1980. Find the last five terms of this finite sequence, which represent the U.S. population for the years 1999 to 2003. (Source: U.S. Census Bureau)

Solution

The last five terms of this finite sequence are as follows.

$$a_{19} = 226.9 + 2.05(19) + 0.035(19)^2 \approx 278.5 \qquad \text{1999 population}$$

$$a_{20} = 226.9 + 2.05(20) + 0.035(20)^2 = 281.9 \qquad \text{2000 population}$$

$$a_{21} = 226.9 + 2.05(21) + 0.035(21)^2 \approx 285.4 \qquad \text{2001 population}$$

$$a_{22} = 226.9 + 2.05(22) + 0.035(22)^2 \approx 288.9 \qquad \text{2002 population}$$

$$a_{23} = 226.9 + 2.05(23) + 0.035(23)^2 \approx 292.6 \qquad \text{2003 population}$$

CHECKPOINT Now try Exercise 111.

Activities

1. Write the first five terms of the sequence. (Assume that n begins with 1.)

   $a_n = \dfrac{2n - 1}{2n}$

   Answer: $\dfrac{1}{2}, \dfrac{3}{4}, \dfrac{5}{6}, \dfrac{7}{8}, \dfrac{9}{10}$

2. Write an expression for the apparent nth term of the sequence

   $0, \dfrac{1}{2}, \dfrac{2}{6}, \dfrac{3}{24}, \dfrac{4}{120}, \ldots$

   Answer: $\dfrac{n - 1}{n!}$

3. Find the sum.

   $\displaystyle\sum_{k=1}^{4} (-1)^k 2k$

   Answer: 4

Exploration

A $3 \times 3 \times 3$ cube is created using 27 unit cubes (a unit cube has a length, width, and height of 1 unit) and only the faces of each cube that are visible are painted blue (see Figure 9.2). Complete the table below to determine how many unit cubes of the $3 \times 3 \times 3$ cube have 0 blue faces, 1 blue face, 2 blue faces, and 3 blue faces. Do the same for a $4 \times 4 \times 4$ cube, a $5 \times 5 \times 5$ cube, and a $6 \times 6 \times 6$ cube and add your results to the table below. What type of pattern do you observe in the table? Write a formula you could use to determine the column values for an $n \times n \times n$ cube.

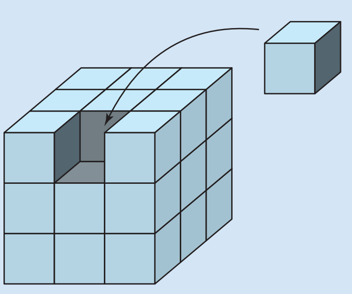


FIGURE 9.2

Number of blue cube faces	0	1	2	3
$3 \times 3 \times 3$				

9.1 Exercises

The *HM mathSpace®* CD-ROM and *Eduspace®* for this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

VOCABULARY CHECK: Fill in the blanks.

1. An ________ ________ is a function whose domain is the set of positive integers.
2. The function values $a_1, a_2, a_3, a_4, \ldots$ are called the ________ of a sequence.
3. A sequence is a ________ sequence if the domain of the function consists of the first n positive integers.
4. If you are given one or more of the first few terms of a sequence, and all other terms of the sequence are defined using previous terms, then the sequence is said to be defined ________.
5. If n is a positive integer, n ________ is defined as $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$.
6. The notation used to represent the sum of the terms of a finite sequence is ________ ________ or sigma notation.
7. For the sum $\sum_{i=1}^{n} a_i$, i is called the ________ of summation, n is the ________ limit of summation, and 1 is the ________ limit of summation.
8. The sum of the terms of a finite or infinite sequence is called a ________.
9. The ________ ________ ________ of a sequence is the sum of the first n terms of the sequence.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–22, write the first five terms of the sequence. (Assume that n begins with 1.)

1. $a_n = 3n + 1$
2. $a_n = 5n - 3$
3. $a_n = 2^n$
4. $a_n = \left(\frac{1}{2}\right)^n$
5. $a_n = (-2)^n$
6. $a_n = \left(-\frac{1}{2}\right)^n$
7. $a_n = \dfrac{n+2}{n}$
8. $a_n = \dfrac{n}{n+2}$
9. $a_n = \dfrac{6n}{3n^2 - 1}$
10. $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$
11. $a_n = \dfrac{1 + (-1)^n}{n}$
12. $a_n = 1 + (-1)^n$
13. $a_n = 2 - \dfrac{1}{3^n}$
14. $a_n = \dfrac{2^n}{3^n}$
15. $a_n = \dfrac{1}{n^{3/2}}$
16. $a_n = \dfrac{10}{n^{2/3}}$
17. $a_n = \dfrac{(-1)^n}{n^2}$
18. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$
19. $a_n = \frac{2}{3}$
20. $a_n = 0.3$
21. $a_n = n(n-1)(n-2)$
22. $a_n = n(n^2 - 6)$

In Exercises 23–26, find the indicated term of the sequence.

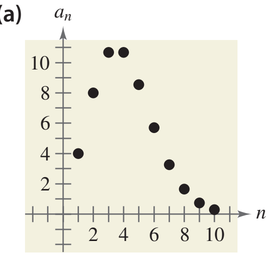
23. $a_n = (-1)^n(3n - 2)$
    $a_{25} =$
24. $a_n = (-1)^{n-1}[n(n-1)]$
    $a_{16} =$
25. $a_n = \dfrac{4n}{2n^2 - 3}$
    $a_{11} =$
26. $a_n = \dfrac{4n^2 - n + 3}{n(n-1)(n+2)}$
    $a_{13} =$

In Exercises 27–32, use a graphing utility to graph the first 10 terms of the sequence. (Assume that n begins with 1.)

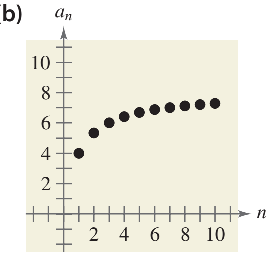
27. $a_n = \dfrac{3}{4}n$
28. $a_n = 2 - \dfrac{4}{n}$
29. $a_n = 16(-0.5)^{n-1}$
30. $a_n = 8(0.75)^{n-1}$
31. $a_n = \dfrac{2n}{n+1}$
32. $a_n = \dfrac{n^2}{n^2 + 2}$

In Exercises 33–36, match the sequence with the graph of its first 10 terms. [The graphs are labeled (a), (b), (c), and (d).]

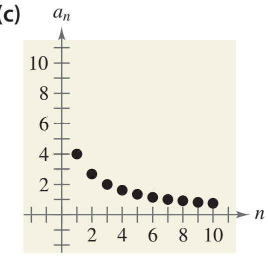
(a)



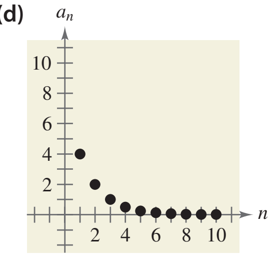
(b)



(c)



(d)



33. $a_n = \dfrac{8}{n+1}$
34. $a_n = \dfrac{8n}{n+1}$
35. $a_n = 4(0.5)^{n-1}$
36. $a_n = \dfrac{4^n}{n!}$

In Exercises 37–50, write an expression for the apparent *n*th term of the sequence. (Assume that *n* begins with 1.)

37. $1, 4, 7, 10, 13, \ldots$

38. $3, 7, 11, 15, 19, \ldots$

39. $0, 3, 8, 15, 24, \ldots$

40. $2, -4, 6, -8, 10, \ldots$

41. $\frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, \frac{-6}{7}, \ldots$

42. $\frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \ldots$

43. $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

44. $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \ldots$

45. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

46. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

47. $1, -1, 1, -1, 1, \ldots$

48. $1, 2, \frac{2^2}{2}, \frac{2^3}{6}, \frac{2^4}{24}, \frac{2^5}{120}, \ldots$

49. $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

50. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

In Exercises 51–54, write the first five terms of the sequence defined recursively.

51. $a_1 = 28, \quad a_{k+1} = a_k - 4$

52. $a_1 = 15, \quad a_{k+1} = a_k + 3$

53. $a_1 = 3, \quad a_{k+1} = 2(a_k - 1)$

54. $a_1 = 32, \quad a_{k+1} = \frac{1}{2}a_k$

In Exercises 55–58, write the first five terms of the sequence defined recursively. Use the pattern to write the *n*th term of the sequence as a function of *n*. (Assume that *n* begins with 1.)

55. $a_1 = 6, \quad a_{k+1} = a_k + 2$

56. $a_1 = 25, \quad a_{k+1} = a_k - 5$

57. $a_1 = 81, \quad a_{k+1} = \frac{1}{3}a_k$

58. $a_1 = 14, \quad a_{k+1} = (-2)a_k$

In Exercises 59–64, write the first five terms of the sequence. (Assume that *n* begins with 0.)

59. $a_n = \dfrac{3^n}{n!}$

60. $a_n = \dfrac{n!}{n}$

61. $a_n = \dfrac{1}{(n+1)!}$

62. $a_n = \dfrac{n^2}{(n+1)!}$

63. $a_n = \dfrac{(-1)^{2n}}{(2n)!}$

64. $a_n = \dfrac{(-1)^{2n+1}}{(2n+1)!}$

In Exercises 65–72, simplify the factorial expression.

65. $\dfrac{4!}{6!}$

66. $\dfrac{5!}{8!}$

67. $\dfrac{10!}{8!}$

68. $\dfrac{25!}{23!}$

69. $\dfrac{(n+1)!}{n!}$

70. $\dfrac{(n+2)!}{n!}$

71. $\dfrac{(2n-1)!}{(2n+1)!}$

72. $\dfrac{(3n+1)!}{(3n)!}$

In Exercises 73–84, find the sum.

73. $\displaystyle\sum_{i=1}^{5}(2i+1)$

74. $\displaystyle\sum_{i=1}^{6}(3i-1)$

75. $\displaystyle\sum_{k=1}^{4}10$

76. $\displaystyle\sum_{k=1}^{5}5$

77. $\displaystyle\sum_{i=0}^{4}i^2$

78. $\displaystyle\sum_{i=0}^{5}2i^2$

79. $\displaystyle\sum_{k=0}^{3}\frac{1}{k^2+1}$

80. $\displaystyle\sum_{j=3}^{5}\frac{1}{j^2-3}$

81. $\displaystyle\sum_{k=2}^{5}(k+1)^2(k-3)$

82. $\displaystyle\sum_{i=1}^{4}[(i-1)^2+(i+1)^3]$

83. $\displaystyle\sum_{i=1}^{4}2^i$

84. $\displaystyle\sum_{j=0}^{4}(-2)^j$

In Exercises 85–88, use a calculator to find the sum.

85. $\displaystyle\sum_{j=1}^{6}(24-3j)$

86. $\displaystyle\sum_{j=1}^{10}\frac{3}{j+1}$

87. $\displaystyle\sum_{k=0}^{4}\frac{(-1)^k}{k+1}$

88. $\displaystyle\sum_{k=0}^{4}\frac{(-1)^k}{k!}$

In Exercises 89–98, use sigma notation to write the sum.

89. $\dfrac{1}{3(1)} + \dfrac{1}{3(2)} + \dfrac{1}{3(3)} + \cdots + \dfrac{1}{3(9)}$

90. $\dfrac{5}{1+1} + \dfrac{5}{1+2} + \dfrac{5}{1+3} + \cdots + \dfrac{5}{1+15}$

91. $\left[2\left(\frac{1}{8}\right)+3\right] + \left[2\left(\frac{2}{8}\right)+3\right] + \cdots + \left[2\left(\frac{8}{8}\right)+3\right]$

92. $\left[1-\left(\frac{1}{6}\right)^2\right] + \left[1-\left(\frac{2}{6}\right)^2\right] + \cdots + \left[1-\left(\frac{6}{6}\right)^2\right]$

93. $3 - 9 + 27 - 81 + 243 - 729$

94. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{128}$

95. $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots - \dfrac{1}{20^2}$

96. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{2 \cdot 4} + \dfrac{1}{3 \cdot 5} + \cdots + \dfrac{1}{10 \cdot 12}$

97. $\frac{1}{4} + \frac{3}{8} + \frac{7}{16} + \frac{15}{32} + \frac{31}{64}$

98. $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \frac{720}{64}$

In Exercises 99–102, find the indicated partial sum of the series.

99. $\displaystyle\sum_{i=1}^{\infty}5\left(\frac{1}{2}\right)^i$

Fourth partial sum

100. $\displaystyle\sum_{i=1}^{\infty}2\left(\frac{1}{3}\right)^i$

Fifth partial sum

101. $\displaystyle\sum_{n=1}^{\infty}4\left(-\frac{1}{2}\right)^n$

Third partial sum

102. $\displaystyle\sum_{n=1}^{\infty}8\left(-\frac{1}{4}\right)^n$

Fourth partial sum

In Exercises 103–106, find the sum of the infinite series.

103. $\displaystyle\sum_{i=1}^{\infty} 6\left(\frac{1}{10}\right)^i$

104. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k$

105. $\displaystyle\sum_{k=1}^{\infty} 7\left(\frac{1}{10}\right)^k$

106. $\displaystyle\sum_{i=1}^{\infty} 2\left(\frac{1}{10}\right)^i$

107. ***Compound Interest*** A deposit of \$5000 is made in an account that earns 8% interest compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 5000\left(1 + \frac{0.08}{4}\right)^n, \qquad n = 1, 2, 3, \ldots.$$

(a) Write the first eight terms of this sequence.

(b) Find the balance in this account after 10 years by finding the 40th term of the sequence.

108. ***Compound Interest*** A deposit of \$100 is made each month in an account that earns 12% interest compounded monthly. The balance in the account after n months is given by

$$A_n = 100(101)[(1.01)^n - 1], \quad n = 1, 2, 3, \ldots.$$

(a) Write the first six terms of this sequence.

(b) Find the balance in this account after 5 years by finding the 60th term of the sequence.

(c) Find the balance in this account after 20 years by finding the 240th term of the sequence.

Model It

109. ***Data Analysis: Number of Stores*** The table shows the numbers a_n of Best Buy stores for the years 1998 to 2003. (Source: Best Buy Company, Inc.)

Year	Number of stores, a_n
1998	311
1999	357
2000	419
2001	481
2002	548
2003	608

Model It (continued)

(a) Use the *regression* feature of a graphing utility to find a linear sequence that models the data. Let n represent the year, with $n = 8$ corresponding to 1998.

(b) Use the *regression* feature of a graphing utility to find a quadratic sequence that models the data.

(c) Evaluate the sequences from parts (a) and (b) for $n = 8, 9, \ldots, 13$. Compare these values with those shown in the table. Which model is a better fit for the data? Explain.

(d) Which model do you think would better predict the number of Best Buy stores in the future? Use the model you chose to predict the number of Best Buy stores in 2008.

110. ***Medicine*** The numbers a_n (in thousands) of AIDS cases reported from 1995 to 2003 can be approximated by the model

$$a_n = 0.0457n^3 - 0.352n^2 - 9.05n + 121.4,$$
$$n = 5, 6, \ldots, 13$$

where n is the year, with $n = 5$ corresponding to 1995. (Source: U.S. Centers for Disease Control and Prevention)

(a) Find the terms of this finite sequence. Use the *statistical plotting* feature of a graphing utility to construct a bar graph that represents the sequence.

(b) What does the graph in part (a) say about reported cases of AIDS?

111. ***Federal Debt*** From 1990 to 2003, the federal debt of the United States rose from just over \$3 trillion to almost \$7 trillion. The federal debt a_n (in billions of dollars) from 1990 to 2003 is approximated by the model

$$a_n = 2.7698n^3 - 61.372n^2 + 600.00n + 3102.9,$$
$$n = 0, 1, \ldots, 13$$

where n is the year, with $n = 0$ corresponding to 1990. (Source: U.S. Office of Management and Budget)

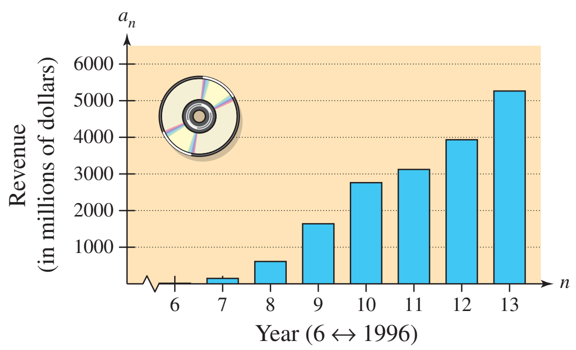
(a) Find the terms of this finite sequence. Use the *statistical plotting* feature of a graphing utility to construct a bar graph that represents the sequence.

(b) What does the pattern in the bar graph in part (a) say about the future of the federal debt?

112. *Revenue* The revenues a_n (in millions of dollars) for Amazon.com for the years 1996 through 2003 are shown in the figure. The revenues can be approximated by the model

$$a_n = 46.609n^2 - 119.84n - 1125.8, \quad n = 6, 7, \ldots, 13$$

where n is the year, with $n = 6$ corresponding to 1996. Use this model to approximate the total revenue from 1996 through 2003. Compare this sum with the result of adding the revenues shown in the figure. (Source: Amazon.com)



Synthesis

True or False? **In Exercises 113 and 114, determine whether the statement is true or false. Justify your answer.**

113. $\displaystyle\sum_{i=1}^{4}(i^2 + 2i) = \sum_{i=1}^{4} i^2 + 2\sum_{i=1}^{4} i$ **114.** $\displaystyle\sum_{j=1}^{4} 2^j = \sum_{j=3}^{6} 2^{j-2}$

Fibonacci Sequence **In Exercises 115 and 116, use the Fibonacci sequence. (See Example 4.)**

115. Write the first 12 terms of the Fibonacci sequence a_n and the first 10 terms of the sequence given by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n \geq 1.$$

116. Using the definition for b_n in Exercise 115, show that b_n can be defined recursively by

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

Arithmetic Mean **In Exercises 117–120, use the following definition of the arithmetic mean $\bar{x}$ of a set of n measurements $x_1, x_2, x_3, \ldots, x_n$.**

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

117. Find the arithmetic mean of the six checking account balances \$327.15, \$785.69, \$433.04, \$265.38, \$604.12, and \$590.30. Use the statistical capabilities of a graphing utility to verify your result.

118. Find the arithmetic mean of the following prices per gallon for regular unleaded gasoline at five gasoline stations in a city: \$1.899, \$1.959, \$1.919, \$1.939, and \$1.999. Use the statistical capabilities of a graphing utility to verify your result.

119. *Proof* Prove that $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x}) = 0.$

120. *Proof* Prove that $\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2.$

In Exercises 121–124, find the first five terms of the sequence.

121. $a_n = \dfrac{x^n}{n!}$

122. $a_n = \dfrac{(-1)^n x^{2n+1}}{2n + 1}$

123. $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$

124. $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n + 1)!}$

Skills Review

In Exercises 125–128, determine whether the function has an inverse function. If it does, find its inverse function.

125. $f(x) = 4x - 3$

126. $g(x) = \dfrac{3}{x}$

127. $h(x) = \sqrt{5x + 1}$

128. $f(x) = (x - 1)^2$

In Exercises 129–132, find (a) $A - B$, (b) $4B - 3A$, (c) AB, and (d) BA.

129. $A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 6 & -3 \end{bmatrix}$

130. $A = \begin{bmatrix} 10 & 7 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -12 \\ 8 & 11 \end{bmatrix}$

131. $A = \begin{bmatrix} -2 & -3 & 6 \\ 4 & 5 & 7 \\ 1 & 7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 6 \\ 0 & 3 & 1 \end{bmatrix}$

132. $A = \begin{bmatrix} -1 & 4 & 0 \\ 5 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

In Exercises 133–136, find the determinant of the matrix.

133. $A = \begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix}$

134. $A = \begin{bmatrix} -2 & 8 \\ 12 & 15 \end{bmatrix}$

135. $A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 7 & 3 \\ 4 & 9 & -1 \end{bmatrix}$

136. $A = \begin{bmatrix} 16 & 11 & 10 & 2 \\ 9 & 8 & 3 & 7 \\ -2 & -1 & 12 & 3 \\ -4 & 6 & 2 & 1 \end{bmatrix}$