### 6.4 Vectors and Dot Products

## What you should learn

- Find the dot product of two vectors and use the Properties of the Dot Product.
- Find the angle between two vectors and determine whether two vectors are orthogonal.
- Write a vector as the sum of two vector components.
- Use vectors to find the work done by a force.


## Why you should learn it

You can use the dot product of two vectors to solve real-life problems involving two vector quantities. For instance, in Exercise 68 on page 468 , you can use the dot product to find the force necessary to keep a sport utility vehicle from rolling down a hill.


## The Dot Product of Two Vectors

So far you have studied two vector operations-vector addition and multiplication by a scalar-each of which yields another vector. In this section, you will study a third vector operation, the dot product. This product yields a scalar, rather than a vector.

## Definition of the Dot Product

The dot product of $\mathbf{u}=\left\langle u_{1}, u_{2}\right\rangle$ and $\mathbf{v}=\left\langle v_{1}, v_{2}\right\rangle$ is

$$
\mathbf{u} \cdot \mathbf{v}=u_{1} v_{1}+u_{2} v_{2} .
$$

## Properties of the Dot Product

Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be vectors in the plane or in space and let $c$ be a scalar.

1. $\mathbf{u} \cdot \mathbf{v}=\mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v}=0$
3. $\mathbf{u} \cdot(\mathbf{v}+\mathbf{w})=\mathbf{u} \cdot \mathbf{v}+\mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v}=\|\mathbf{v}\|^{2}$
5. $c(\mathbf{u} \cdot \mathbf{v})=c \mathbf{u} \cdot \mathbf{v}=\mathbf{u} \cdot c \mathbf{v}$

For proofs of the properties of the dot product, see Proofs in Mathematics on page 492.

## Example 1 Finding Dot Products

Find each dot product.
a. $\langle 4,5\rangle \cdot\langle 2,3\rangle$
b. $\langle 2,-1\rangle \cdot\langle 1,2\rangle$
c. $\langle 0,3\rangle \cdot\langle 4,-2\rangle$

## Solution

a. $\langle 4,5\rangle \cdot\langle 2,3\rangle=4(2)+5(3)$

$$
=8+15
$$

$$
=23
$$

b. $\langle 2,-1\rangle \cdot\langle 1,2\rangle=2(1)+(-1)(2)=2-2=0$
c. $\langle 0,3\rangle \cdot\langle 4,-2\rangle=0(4)+3(-2)=0-6=-6$
©CHECKPOINT Now try Exercise 1.
In Example 1, be sure you see that the dot product of two vectors is a scalar (a real number), not a vector. Moreover, notice that the dot product can be positive, zero, or negative.

## Example 2 Using Properties of Dot Products

Let $\mathbf{u}=\langle-1,3\rangle, \mathbf{v}=\langle 2,-4\rangle$, and $\mathbf{w}=\langle 1,-2\rangle$. Find each dot product.
a. $(\mathbf{u} \cdot \mathbf{v}) \mathbf{w}$
b. $\mathbf{u} \cdot 2 \mathbf{v}$

## Solution

Begin by finding the dot product of $\mathbf{u}$ and $\mathbf{v}$.

$$
\begin{aligned}
\mathbf{u} \cdot \mathbf{v} & =\langle-1,3\rangle \cdot\langle 2,-4\rangle \\
& =(-1)(2)+3(-4) \\
& =-14
\end{aligned}
$$

a. $(\mathbf{u} \cdot \mathbf{v}) \mathbf{w}=-14\langle 1,-2\rangle$

$$
=\langle-14,28\rangle
$$

b. $\mathbf{u} \cdot 2 \mathbf{v}=2(\mathbf{u} \cdot \mathbf{v})$

$$
=2(-14)
$$

$$
=-28
$$

Notice that the product in part (a) is a vector, whereas the product in part (b) is a scalar. Can you see why?

## (CHECKPOINT Now try Exercise 11.

## Example 3 Dot Product and Magnitude

The dot product of $\mathbf{u}$ with itself is 5 . What is the magnitude of $\mathbf{u}$ ?

## Solution

Because $\|\mathbf{u}\|^{2}=\mathbf{u} \cdot \mathbf{u}$ and $\mathbf{u} \cdot \mathbf{u}=5$, it follows that

$$
\begin{aligned}
\|\mathbf{u}\| & =\sqrt{\mathbf{u} \cdot \mathbf{u}} \\
& =\sqrt{5} .
\end{aligned}
$$

dCHECKPOINT Now try Exercise 19.

## The Angle Between Two Vectors



FIGURE 6.33

The angle between two nonzero vectors is the angle $\theta, 0 \leq \theta \leq \pi$, between their respective standard position vectors, as shown in Figure 6.33. This angle can be found using the dot product. (Note that the angle between the zero vector and another vector is not defined.)

Angle Between Two Vectors
If $\theta$ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$
\cos \theta=\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}
$$

For a proof of the angle between two vectors, see Proofs in Mathematics on page 492.


FIGURE 6.34

## Example 4 Finding the Angle Between Two Vectors

Find the angle between $\mathbf{u}=\langle 4,3\rangle$ and $\mathbf{v}=\langle 3,5\rangle$.

## Solution

$$
\begin{aligned}
\cos \theta & =\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \\
& =\frac{\langle 4,3\rangle \cdot\langle 3,5\rangle}{\|\langle 4,3\rangle\|\|\langle 3,5\rangle\|} \\
& =\frac{27}{5 \sqrt{34}}
\end{aligned}
$$

This implies that the angle between the two vectors is

$$
\theta=\arccos \frac{27}{5 \sqrt{34}} \approx 22.2^{\circ}
$$

as shown in Figure 6.34.
(CHECKPOINT Now try Exercise 29.
Rewriting the expression for the angle between two vectors in the form

$$
\mathbf{u} \cdot \mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\| \cos \theta \quad \text { Alternative form of dot product }
$$

produces an alternative way to calculate the dot product. From this form, you can see that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u} \cdot \mathbf{v}$ and $\cos \theta$ will always have the same sign. Figure 6.35 shows the five possible orientations of two vectors.


## Definition of Orthogonal Vectors

The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v}=0$.

The terms orthogonal and perpendicular mean essentially the same thing-meeting at right angles. Even though the angle between the zero vector and another vector is not defined, it is convenient to extend the definition of orthogonality to include the zero vector. In other words, the zero vector is orthogonal to every vector $\mathbf{u}$, because $\mathbf{0} \cdot \mathbf{u}=0$.

## Technology

The graphing utility program Finding the Angle Between Two Vectors, found on our website college.hmco.com, graphs two vectors $\mathbf{u}=\langle a, b\rangle$ and $\mathbf{v}=\langle c, d\rangle$ in standard position and finds the measure of the angle between them. Use the program to verify the solutions for Examples 4 and 5.

## Example 5 Determining Orthogonal Vectors

Are the vectors $\mathbf{u}=\langle 2,-3\rangle$ and $\mathbf{v}=\langle 6,4\rangle$ orthogonal?

## Solution

Begin by finding the dot product of the two vectors.

$$
\mathbf{u} \cdot \mathbf{v}=\langle 2,-3\rangle \cdot\langle 6,4\rangle=2(6)+(-3)(4)=0
$$

Because the dot product is 0 , the two vectors are orthogonal (see Figure 6.36).


FIGURE 6.36

## dCHECKPOINT Now try Exercise 47.

## Finding Vector Components

You have already seen applications in which two vectors are added to produce a resultant vector. Many applications in physics and engineering pose the reverse problem-decomposing a given vector into the sum of two vector components.

Consider a boat on an inclined ramp, as shown in Figure 6.37. The force $\mathbf{F}$ due to gravity pulls the boat down the ramp and against the ramp. These two orthogonal forces, $\mathbf{w}_{1}$ and $\mathbf{w}_{2}$, are vector components of $\mathbf{F}$. That is,

$$
\mathbf{F}=\mathbf{w}_{1}+\mathbf{w}_{2} . \quad \text { Vector components of } \mathbf{F}
$$

The negative of component $\mathbf{w}_{1}$ represents the force needed to keep the boat from rolling down the ramp, whereas $\mathbf{w}_{2}$ represents the force that the tires must withstand against the ramp. A procedure for finding $\mathbf{w}_{1}$ and $\mathbf{w}_{2}$ is shown on the following page.


FIGURE 6.37

## Definition of Vector Components

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors such that
$\mathbf{u}=\mathbf{w}_{1}+\mathbf{w}_{2}$
where $\mathbf{w}_{1}$ and $\mathbf{w}_{2}$ are orthogonal and $\mathbf{w}_{1}$ is parallel to (or a scalar multiple of) $\mathbf{v}$, as shown in Figure 6.38. The vectors $\mathbf{w}_{1}$ and $\mathbf{w}_{2}$ are called vector components of $\mathbf{u}$. The vector $\mathbf{w}_{1}$ is the projection of $\mathbf{u}$ onto $\mathbf{v}$ and is denoted by

$$
\mathbf{w}_{1}=\operatorname{proj}_{\mathbf{v}} \mathbf{u}
$$

The vector $\mathbf{w}_{2}$ is given by $\mathbf{w}_{2}=\mathbf{u}-\mathbf{w}_{1}$.

$\theta$ is acute.
FIGURE 6.38

From the definition of vector components, you can see that it is easy to find the component $\mathbf{w}_{2}$ once you have found the projection of $\mathbf{u}$ onto $\mathbf{v}$. To find the projection, you can use the dot product, as follows.

$$
\begin{aligned}
\mathbf{u} & =\mathbf{w}_{1}+\mathbf{w}_{2}=c \mathbf{v}+\mathbf{w}_{2} & & \mathbf{w}_{1} \text { is a scalar multiple of } \\
\mathbf{u} \cdot \mathbf{v} & =\left(c \mathbf{v}+\mathbf{w}_{2}\right) \cdot \mathbf{v} & & \text { Take dot product of each } \\
& =c \mathbf{v} \cdot \mathbf{v}+\mathbf{w}_{2} \cdot \mathbf{v} & & \\
& =c\|\mathbf{v}\|^{2}+0 & & \mathbf{w}_{2} \text { and } \mathbf{v} \text { are orthogonal. }
\end{aligned}
$$

So,

$$
c=\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^{2}}
$$

and

$$
\mathbf{w}_{1}=\operatorname{proj}_{\mathbf{v}} \mathbf{u}=c \mathbf{v}=\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^{2}} \mathbf{v}
$$

## Projection of $\mathbf{u}$ onto $\mathbf{v}$

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$
\operatorname{proj}_{\mathbf{v}} \mathbf{u}=\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^{2}}\right) \mathbf{v}
$$



FIGURE 6.39

## Example 6 Decomposing a Vector into Components

Find the projection of $\mathbf{u}=\langle 3,-5\rangle$ onto $\mathbf{v}=\langle 6,2\rangle$. Then write $\mathbf{u}$ as the sum of two orthogonal vectors, one of which is $\operatorname{proj}_{v} \mathbf{u}$.

## Solution

The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$
\mathbf{w}_{1}=\operatorname{proj}_{\mathbf{v}} \mathbf{u}=\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^{2}}\right) \mathbf{v}=\left(\frac{8}{40}\right)\langle 6,2\rangle=\left\langle\frac{6}{5}, \frac{2}{5}\right\rangle
$$

as shown in Figure 6.39. The other component, $\mathbf{w}_{2}$, is

$$
\mathbf{w}_{2}=\mathbf{u}-\mathbf{w}_{1}=\langle 3,-5\rangle-\left\langle\frac{6}{5}, \frac{2}{5}\right\rangle=\left\langle\frac{9}{5},-\frac{27}{5}\right\rangle
$$

So,

$$
\mathbf{u}=\mathbf{w}_{1}+\mathbf{w}_{2}=\left\langle\frac{6}{5}, \frac{2}{5}\right\rangle+\left\langle\frac{9}{5},-\frac{27}{5}\right\rangle=\langle 3,-5\rangle
$$

dCHECKPOINT Now try Exercise 53.

## Example 7 Finding a Force

A 200-pound cart sits on a ramp inclined at $30^{\circ}$, as shown in Figure 6.40. What force is required to keep the cart from rolling down the ramp?

## Solution

Because the force due to gravity is vertical and downward, you can represent the gravitational force by the vector

$$
\mathbf{F}=-200 \mathbf{j} . \quad \text { Force due to gravity }
$$

To find the force required to keep the cart from rolling down the ramp, project $\mathbf{F}$ onto a unit vector $\mathbf{v}$ in the direction of the ramp, as follows.

$$
\mathbf{v}=\left(\cos 30^{\circ}\right) \mathbf{i}+\left(\sin 30^{\circ}\right) \mathbf{j}=\frac{\sqrt{3}}{2} \mathbf{i}+\frac{1}{2} \mathbf{j} \quad \text { Unit vector along ramp }
$$

Therefore, the projection of $\mathbf{F}$ onto $\mathbf{v}$ is

$$
\begin{aligned}
\mathbf{w}_{1} & =\operatorname{proj}_{\mathbf{v}} \mathbf{F} \\
& =\left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^{2}}\right) \mathbf{v} \\
& =(\mathbf{F} \cdot \mathbf{v}) \mathbf{v} \\
& =(-200)\left(\frac{1}{2}\right) \mathbf{v} \\
& =-100\left(\frac{\sqrt{3}}{2} \mathbf{i}+\frac{1}{2} \mathbf{j}\right)
\end{aligned}
$$

The magnitude of this force is 100 , and so a force of 100 pounds is required to keep the cart from rolling down the ramp.
dCHEKKPOINT Now try Exercise 67.

## Activities

1. Find the dot product.
$\langle-3,6\rangle \cdot\langle 1,0\rangle$
Answer: - 3
2. Are the vectors $\mathbf{u}=\langle 5,-2\rangle$ and $\mathbf{v}=\langle 3,4\rangle$ orthogonal? Answer: No
3. A person pulls a wagon with a constant force of 15 pounds at a constant angle of $40^{\circ}$ for 500 feet. Find the work done in pulling the wagon.
Answer: $\approx 5745$ foot-pounds


FIGURE 6.43

## Work

The work $W$ done by a constant force $\mathbf{F}$ acting along the line of motion of an object is given by

$$
W=(\text { magnitude of force })(\text { distance })=\|\mathbf{F}\|\|\stackrel{\rightharpoonup}{P Q}\|
$$

as shown in Figure 6.41. If the constant force $\mathbf{F}$ is not directed along the line of motion, as shown in Figure 6.42, the work $W$ done by the force is given by

$$
\begin{aligned}
W & =\|\operatorname{proj} \stackrel{\rightharpoonup}{P Q} \mathbf{F}\|\|\stackrel{\rightharpoonup}{P Q}\| \\
& =(\cos \theta)\|\mathbf{F}\|\|\stackrel{\rightharpoonup}{P Q}\| \\
& =\mathbf{F} \cdot \stackrel{\rightharpoonup}{P Q}
\end{aligned}
$$

$$
\begin{aligned}
& \text { Projection form for work } \\
& \left\|\operatorname{proj}_{\overrightarrow{P Q}} \mathbf{F}\right\|=(\cos \theta)\|\mathbf{F}\| \\
& \text { Alternative form of dot product }
\end{aligned}
$$



Force acts along the line of motion. FIGURE 6.41


Force acts at angle $\theta$ with the line of motion. FIGURE 6.42

This notion of work is summarized in the following definition.

## Definition of Work

The work $W$ done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{P Q}$ is given by either of the following.

1. $W=\left\|\operatorname{proj}_{P Q} \stackrel{F}{ }\right\|\|\stackrel{\rightharpoonup}{P Q}\| \quad$ Projection form
2. $W=\mathbf{F} \cdot \overrightarrow{P Q} \quad$ Dot product form

## Example 8 Finding Work

To close a sliding door, a person pulls on a rope with a constant force of 50 pounds at a constant angle of $60^{\circ}$, as shown in Figure 6.43. Find the work done in moving the door 12 feet to its closed position.

## Solution

Using a projection, you can calculate the work as follows.

$$
\begin{aligned}
W & =\|\operatorname{proj} \stackrel{\rightharpoonup}{P Q} \mathbf{F}\|\|\stackrel{\rightharpoonup}{P Q}\| \quad \text { Projection form for work } \\
& =\left(\cos 60^{\circ}\right)\|\mathbf{F}\|\|\stackrel{\rightharpoonup}{P Q}\| \\
& =\frac{1}{2}(50)(12)=300 \text { foot-pounds }
\end{aligned}
$$

So, the work done is 300 foot-pounds. You can verify this result by finding the vectors $\mathbf{F}$ and $\stackrel{\rightharpoonup}{P Q}$ and calculating their dot product.
©CHECKPOINT Now try Exercise 69.

### 6.4 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The $\qquad$ of two vectors yields a scalar, rather than a vector.
2. If $\theta$ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then $\cos \theta=$ $\qquad$ .
3. The vectors $\mathbf{u}$ and $\mathbf{v}$ are $\qquad$ if $\mathbf{u} \cdot \mathbf{v}=0$.
4. The projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by $\operatorname{proj}_{\mathbf{v}} \mathbf{u}=$ $\qquad$ .
5. The work $W$ done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\stackrel{\rightharpoonup}{P Q}$ is given by $W=$ $\qquad$ or $W=$ $\qquad$ —.
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at www.Eduspace.com.

## In Exercises 1-8, find the dot product of $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u}=\langle 6,1\rangle$
$\mathbf{v}=\langle-2,3\rangle$
2. $\mathbf{u}=\langle 5,12\rangle$
$\mathbf{v}=\langle-3,2\rangle$
3. $\mathbf{u}=\langle-4,1\rangle$
4. $\mathbf{u}=\langle-2,5\rangle$
$\mathbf{v}=\langle-1,-2\rangle$
$\mathbf{v}=\langle 2,-3\rangle$
5. $\mathbf{u}=3 \mathbf{i}+4 \mathbf{j}$
$\mathbf{v}=\mathbf{i}-\mathbf{j}$
$\mathbf{v}=7 \mathbf{i}-2 \mathbf{j}$
6. $\mathbf{u}=3 \mathbf{i}+2 \mathbf{j}$
$\mathbf{v}=-2 \mathbf{i}-3 \mathbf{j}$
7. $\mathbf{u}=\mathbf{i}-2 \mathbf{j}$
$\mathbf{v}=-2 \mathbf{i}+\mathbf{j}$

In Exercises 9-18, use the vectors $\mathbf{u}=\langle 2,2\rangle, \mathbf{v}=\langle-3,4\rangle$, and $w=\langle 1,-2\rangle$ to find the indicated quantity. State whether the result is a vector or a scalar.
9. $\mathbf{u} \cdot \mathbf{u}$
10. $3 \mathbf{u} \cdot \mathrm{v}$
11. $(\mathbf{u} \cdot \mathbf{v}) \mathbf{v}$
12. $(\mathbf{v} \cdot \mathbf{u}) \mathbf{W}$
13. $(3 \mathbf{w} \cdot \mathbf{v}) \mathbf{u}$
14. $(u \cdot 2 v) w$
15. $\|\mathbf{w}\|-1$
16. $2-\|\mathbf{u}\|$
17. $(\mathbf{u} \cdot \mathbf{v})-(\mathbf{u} \cdot \mathbf{w})$
18. $(\mathbf{v} \cdot \mathbf{u})-(\mathbf{w} \cdot \mathbf{v})$

In Exercises 19-24, use the dot product to find the magnitude of $\mathbf{u}$.
19. $\mathbf{u}=\langle-5,12\rangle$
20. $\mathbf{u}=\langle 2,-4\rangle$
21. $\mathbf{u}=20 \mathbf{i}+25 \mathbf{j}$
22. $\mathbf{u}=12 \mathbf{i}-16 \mathbf{j}$
23. $\mathbf{u}=6 \mathbf{j}$
24. $\mathbf{u}=-21 \mathbf{i}$

In Exercises 25-34, find the angle $\boldsymbol{\theta}$ between the vectors.
25. $\mathbf{u}=\langle 1,0\rangle$
26. $\mathbf{u}=\langle 3,2\rangle$
$\mathbf{v}=\langle 0,-2\rangle$
$\mathbf{v}=\langle 4,0\rangle$
27. $\mathbf{u}=3 \mathbf{i}+4 \mathbf{j}$

$$
\mathbf{v}=-2 \mathbf{j}
$$

28. $\mathbf{u}=2 \mathbf{i}-3 \mathbf{j}$
$\mathbf{v}=\mathbf{i}-2 \mathbf{j}$
29. $\mathbf{u}=2 \mathbf{i}-\mathbf{j}$
$\mathbf{v}=6 \mathbf{i}+4 \mathbf{j}$
30. $\mathbf{u}=-6 \mathbf{i}-3 \mathbf{j}$
$\mathbf{v}=-8 \mathbf{i}+4 \mathbf{j}$
31. $\begin{aligned} \mathbf{u} & =5 \mathbf{i}+5 \mathbf{j} \\ \mathbf{v} & =-6 \mathbf{i}+6 \mathbf{j}\end{aligned}$
32. $\mathbf{u}=2 \mathbf{i}-3 \mathbf{j}$

$$
\mathbf{v}=4 \mathbf{i}+3 \mathbf{j}
$$

33. $\mathbf{u}=\cos \left(\frac{\pi}{3}\right) \mathbf{i}+\sin \left(\frac{\pi}{3}\right) \mathbf{j}$

$$
\mathbf{v}=\cos \left(\frac{3 \pi}{4}\right) \mathbf{i}+\sin \left(\frac{3 \pi}{4}\right) \mathbf{j}
$$

34. $\mathbf{u}=\cos \left(\frac{\pi}{4}\right) \mathbf{i}+\sin \left(\frac{\pi}{4}\right) \mathbf{j}$

$$
\mathbf{v}=\cos \left(\frac{\pi}{2}\right) \mathbf{i}+\sin \left(\frac{\pi}{2}\right) \mathbf{j}
$$

In Exercises 35-38, graph the vectors and find the degree measure of the angle $\boldsymbol{\theta}$ between the vectors.
35. $\mathbf{u}=3 \mathbf{i}+4 \mathbf{j}$
36. $\mathbf{u}=6 \mathbf{i}+3 \mathbf{j}$
$\mathbf{v}=-7 \mathbf{i}+5 \mathbf{j}$
$\mathbf{v}=-4 \mathbf{i}+4 \mathbf{j}$
37. $\mathbf{u}=5 \mathbf{i}+5 \mathbf{j}$
$\mathbf{v}=-8 \mathbf{i}+8 \mathbf{j}$
38. $\mathbf{u}=2 \mathbf{i}-3 \mathbf{j}$
$\mathbf{v}=8 \mathbf{i}+3 \mathbf{j}$

In Exercises 39-42, use vectors to find the interior angles of the triangle with the given vertices.
39. $(1,2),(3,4),(2,5)$
40. $(-3,-4),(1,7),(8,2)$
41. $(-3,0),(2,2),(0,6)$
42. $(-3,5),(-1,9),(7,9)$

In Exercises 43-46, find $\mathbf{u} \cdot \mathbf{v}$, where $\boldsymbol{\theta}$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
43. $\|\mathbf{u}\|=4,\|\mathbf{v}\|=10, \theta=\frac{2 \pi}{3}$
44. $\|\mathbf{u}\|=100,\|\mathbf{v}\|=250, \quad \theta=\frac{\pi}{6}$
45. $\|\mathbf{u}\|=9,\|\mathbf{v}\|=36, \quad \theta=\frac{3 \pi}{4}$
46. $\|\mathbf{u}\|=4,\|\mathbf{v}\|=12, \quad \theta=\frac{\pi}{3}$

In Exercises 47-52, determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, parallel, or neither.
47. $\mathbf{u}=\langle-12,30\rangle$
48. $\mathbf{u}=\langle 3,15\rangle$
$\mathbf{v}=\left\langle\frac{1}{2},-\frac{5}{4}\right\rangle$
$\mathbf{v}=\langle-1,5\rangle$
49. $\mathbf{u}=\frac{1}{4}(3 \mathbf{i}-\mathbf{j})$
50. $\mathbf{u}=\mathbf{i}$
$\mathbf{v}=5 \mathbf{i}+6 \mathbf{j}$

$$
\mathbf{v}=-2 \mathbf{i}+2 \mathbf{j}
$$

51. $\mathbf{u}=2 \mathbf{i}-2 \mathbf{j}$
52. $\mathbf{u}=\langle\cos \theta, \sin \theta\rangle$
$\mathbf{v}=-\mathbf{i}-\mathbf{j}$

$$
\mathbf{v}=\langle\sin \theta,-\cos \theta\rangle
$$

In Exercises 53-56, find the projection of $\mathbf{u}$ onto $\mathbf{v}$. Then write $\mathbf{u}$ as the sum of two orthogonal vectors, one of which is $\operatorname{proj}_{\mathrm{v}} \mathbf{u}$.
53. $\mathbf{u}=\langle 2,2\rangle$
54. $\mathbf{u}=\langle 4,2\rangle$
$\mathbf{v}=\langle 6,1\rangle$
$\mathbf{v}=\langle 1,-2\rangle$
55. $\mathbf{u}=\langle 0,3\rangle$
56. $\mathbf{u}=\langle-3,-2\rangle$
$\mathbf{v}=\langle 2,15\rangle$
$\mathbf{v}=\langle-4,-1\rangle$

In Exercises 57 and 58, use the graph to determine mentally the projection of $\mathbf{u}$ onto $\mathbf{v}$. (The coordinates of the terminal points of the vectors in standard position are given.) Use the formula for the projection of $\mathbf{u}$ onto $\mathbf{v}$ to verify your result.
57.

58.


In Exercises 59-62, find two vectors in opposite directions that are orthogonal to the vector $\mathbf{u}$. (There are many correct answers.)
59. $\mathbf{u}=\langle 3,5\rangle$
60. $\mathbf{u}=\langle-8,3\rangle$
61. $\mathbf{u}=\frac{1}{2} \mathbf{i}-\frac{2}{3} \mathbf{j}$
62. $\mathbf{u}=-\frac{5}{2} \mathbf{i}-3 \mathbf{j}$

Work In Exercises 63 and 64, find the work done in moving a particle from $P$ to $Q$ if the magnitude and direction of the force are given by $\mathbf{v}$.
63. $P=(0,0), \quad Q=(4,7), \quad \mathbf{v}=\langle 1,4\rangle$
64. $P=(1,3), \quad Q=(-3,5), \quad \mathbf{v}=-2 \mathbf{i}+3 \mathbf{j}$
65. Revenue The vector $\mathbf{u}=\langle 1650,3200\rangle$ gives the numbers of units of two types of baking pans produced by a company. The vector $\mathbf{v}=\langle 15.25,10.50\rangle$ gives the prices (in dollars) of the two types of pans, respectively.
(a) Find the dot product $\mathbf{u} \cdot \mathbf{v}$ and interpret the result in the context of the problem.
(b) Identify the vector operation used to increase the prices by $5 \%$.
66. Revenue The vector $\mathbf{u}=\langle 3240,2450\rangle$ gives the numbers of hamburgers and hot dogs, respectively, sold at a fast-food stand in one month. The vector $\mathbf{v}=\langle 1.75,1.25\rangle$ gives the prices (in dollars) of the food items.
(a) Find the dot product $\mathbf{u} \cdot \mathbf{v}$ and interpret the result in the context of the problem.
(b) Identify the vector operation used to increase the prices by $2.5 \%$.

## Model It

67. Braking Load A truck with a gross weight of 30,000 pounds is parked on a slope of $d^{\circ}$ (see figure). Assume that the only force to overcome is the force of gravity.

(a) Find the force required to keep the truck from rolling down the hill in terms of the slope $d$.
(b) Use a graphing utility to complete the table.

| $d$ | $0^{\circ}$ | $1^{\circ}$ | $2^{\circ}$ | $3^{\circ}$ | $4^{\circ}$ | $5^{\circ}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Force |  |  |  |  |  |  |


| $d$ | $6^{\circ}$ | $7^{\circ}$ | $8^{\circ}$ | $9^{\circ}$ | $10^{\circ}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Force |  |  |  |  |  |

(c) Find the force perpendicular to the hill when $d=5^{\circ}$.
68. Braking Load A sport utility vehicle with a gross weight of 5400 pounds is parked on a slope of $10^{\circ}$. Assume that the only force to overcome is the force of gravity. Find the force required to keep the vehicle from rolling down the hill. Find the force perpendicular to the hill.
69. Work Determine the work done by a person lifting a 25-kilogram (245-newton) bag of sugar.
70. Work Determine the work done by a crane lifting a 2400-pound car 5 feet.
71. Work A force of 45 pounds exerted at an angle of $30^{\circ}$ above the horizontal is required to slide a table across a floor (see figure). The table is dragged 20 feet. Determine the work done in sliding the table.

72. Work A tractor pulls a $\log 800$ meters, and the tension in the cable connecting the tractor and log is approximately 1600 kilograms (15,691 newtons). The direction of the force is $35^{\circ}$ above the horizontal. Approximate the work done in pulling the log.
73. Work One of the events in a local strongman contest is to pull a cement block 100 feet. One competitor pulls the block by exerting a force of 250 pounds on a rope attached to the block at an angle of $30^{\circ}$ with the horizontal (see figure). Find the work done in pulling the block.

74. Work A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a $20^{\circ}$ angle with the horizontal (see figure). Find the work done in pulling the wagon 50 feet.


## Synthesis

True or False? In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.
75. The work $W$ done by a constant force $\mathbf{F}$ acting along the line of motion of an object is represented by a vector.
76. A sliding door moves along the line of vector $\overrightarrow{P Q}$. If a force is applied to the door along a vector that is orthogonal to $\overrightarrow{P Q}$, then no work is done.
77. Think About It What is known about $\theta$, the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, under each condition?
(a) $\mathbf{u} \cdot \mathbf{v}=0$
(b) $\mathbf{u} \cdot \mathbf{v}>0$
(c) $\mathbf{u} \cdot \mathbf{v}<0$
78. Think About It What can be said about the vectors $\mathbf{u}$ and $\mathbf{v}$ under each condition?
(a) The projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{u}$.
(b) The projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{0}$.
79. Proof Use vectors to prove that the diagonals of a rhombus are perpendicular.
80. Proof Prove the following.
$\|\mathbf{u}-\mathbf{v}\|^{2}=\|\mathbf{u}\|^{2}+\|\mathbf{v}\|^{2}-2 \mathbf{u} \cdot \mathbf{v}$

## Skills Review

In Exercises 81-84, perform the operation and write the result in standard form.
81. $\sqrt{42} \cdot \sqrt{24}$
82. $\sqrt{18} \cdot \sqrt{112}$
83. $\sqrt{-3} \cdot \sqrt{-8}$
84. $\sqrt{-12} \cdot \sqrt{-96}$

In Exercises 85-88, find all solutions of the equation in the interval $[0,2 \pi)$.
85. $\sin 2 x-\sqrt{3} \sin x=0$
86. $\sin 2 x+\sqrt{2} \cos x=0$
87. $2 \tan x=\tan 2 x$
88. $\cos 2 x-3 \sin x=2$

In Exercises 89-92, find the exact value of the trigonometric function given that $\sin u=-\frac{12}{13}$ and $\cos v=\frac{24}{25}$. (Both $u$ and $v$ are in Quadrant IV.)
89. $\sin (u-v)$
90. $\sin (u+v)$
91. $\cos (v-u)$
92. $\tan (u-v)$

