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6.2 Law of Cosines

What you should learn

- Use the Law of Cosines to solve oblique triangles (SSS or SAS).
- Use the Law of Cosines to model and solve real-life problems.
- Use Heron's Area Formula to find the area of a triangle.

Why you should learn it

You can use the Law of Cosines to solve real-life problems involving oblique triangles. For instance, in Exercise 31 on page 444, you can use the Law of Cosines to approximate the length of a marsh.

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In cases where the Law of Cosines must be used, encourage your students to solve for the largest angle first, then finish the problem using either the Law of Sines or the Law of Cosines.

Introduction

Two cases remain in the list of conditions needed to solve an oblique triangle—SSS and SAS. If you are given three sides (SSS), or two sides and their included angle (SAS), none of the ratios in the Law of Sines would be complete. In such cases, you can use the **Law of Cosines.**

Law of Cosines

Standard Form	*Alternative Form*
$a^2 = b^2 + c^2 - 2bc\cos A$	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$b^2 = a^2 + c^2 - 2ac\cos B$	$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$c^2 = a^2 + b^2 - 2ab\cos C$	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

For a proof of the Law of Cosines, see Proofs in Mathematics on page 490.

Example 1 Three Sides of a Triangle—SSS

Find the three angles of the triangle in Figure 6.11.

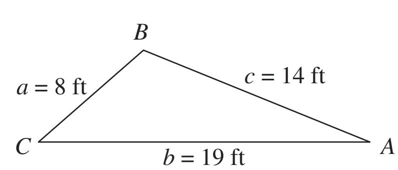

FIGURE 6.11

Solution

It is a good idea first to find the angle opposite the longest side—side b in this case. Using the alternative form of the Law of Cosines, you find that

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{8^2 + 14^2 - 19^2}{2(8)(14)} \approx -0.45089.$$

Because $\cos B$ is negative, you know that B is an *obtuse* angle given by $B \approx 116.80°$. At this point, it is simpler to use the Law of Sines to determine A.

$$\sin A = a\left(\frac{\sin B}{b}\right) \approx 8\left(\frac{\sin 116.80°}{19}\right) \approx 0.37583$$

Because B is obtuse, A must be acute, because a triangle can have, at most, one obtuse angle. So, $A \approx 22.08°$ and $C \approx 180° - 22.08° - 116.80° = 41.12°$.

CHECKPOINT Now try Exercise 1.

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Exploration

What familiar formula do you obtain when you use the third form of the Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$

and you let $C = 90°$? What is the relationship between the Law of Cosines and this formula?

Do you see why it was wise to find the largest angle *first* in Example 1? Knowing the cosine of an angle, you can determine whether the angle is acute or obtuse. That is,

$\cos \theta > 0$ for $0° < \theta < 90°$ Acute

$\cos \theta < 0$ for $90° < \theta < 180°$. Obtuse

So, in Example 1, once you found that angle B was obtuse, you knew that angles A and C were both acute. If the largest angle is acute, the remaining two angles are acute also.

Example 2 Two Sides and the Included Angle—SAS

Find the remaining angles and side of the triangle in Figure 6.12.

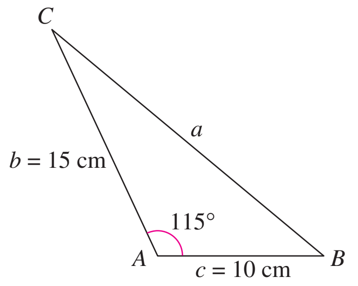

FIGURE 6.12

Solution

Use the Law of Cosines to find the unknown side a in the figure.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 15^2 + 10^2 - 2(15)(10) \cos 115°$$

$$a^2 \approx 451.79$$

$$a \approx 21.26$$

Because $a \approx 21.26$ centimeters, you now know the ratio $\sin A/a$ and you can use the reciprocal form of the Law of Sines to solve for B.

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = b\left(\frac{\sin A}{a}\right)$$

$$= 15\left(\frac{\sin 115°}{21.26}\right)$$

$$\approx 0.63945$$

So, $B = \arcsin 0.63945 \approx 39.75°$ and $C \approx 180° - 115° - 39.75° = 25.25°$.

CHECKPOINT Now try Exercise 3.

STUDY TIP

When solving an oblique triangle given three sides, you use the alternative form of the Law of Cosines to solve for an angle. When solving an oblique triangle given two sides and their included angle, you use the standard form of the Law of Cosines to solve for an unknown.

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Applications

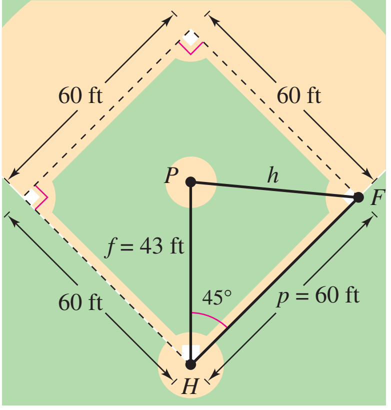

FIGURE 6.13

Example 3 An Application of the Law of Cosines

The pitcher's mound on a women's softball field is 43 feet from home plate and the distance between the bases is 60 feet, as shown in Figure 6.13. (The pitcher's mound is not halfway between home plate and second base.) How far is the pitcher's mound from first base?

Solution

In triangle HPF, $H = 45°$ (line HP bisects the right angle at H), $f = 43$, and $p = 60$. Using the Law of Cosines for this SAS case, you have

$$h^2 = f^2 + p^2 - 2fp \cos H$$

$$= 43^2 + 60^2 - 2(43)(60) \cos 45° \approx 1800.3$$

So, the approximate distance from the pitcher's mound to first base is

$$h \approx \sqrt{1800.3} \approx 42.43 \text{ feet.}$$

CHECKPOINT Now try Exercise 31.

Example 4 An Application of the Law of Cosines

A ship travels 60 miles due east, then adjusts its course northward, as shown in Figure 6.14. After traveling 80 miles in that direction, the ship is 139 miles from its point of departure. Describe the bearing from point B to point C.

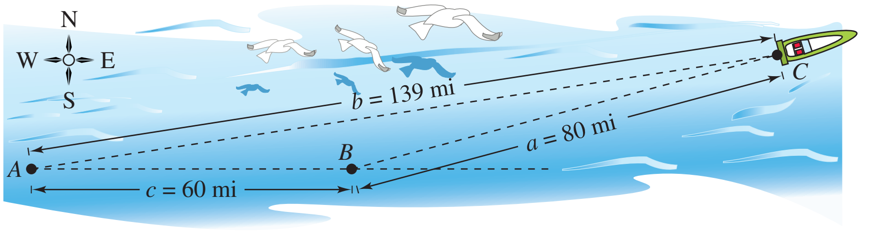

FIGURE 6.14

Solution

You have $a = 80$, $b = 139$, and $c = 60$; so, using the alternative form of the Law of Cosines, you have

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{80^2 + 60^2 - 139^2}{2(80)(60)}$$

$$\approx -0.97094.$$

So, $B \approx \arccos(-0.97094) \approx 166.15°$, and thus the bearing measured from due north from point B to point C is $166.15° - 90° = 76.15°$, or N 76.15° E.

CHECKPOINT Now try Exercise 37.

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Historical Note

Heron of Alexandria (c. 100 B.C.) was a Greek geometer and inventor. His works describe how to find the areas of triangles, quadrilaterals, regular polygons having 3 to 12 sides, and circles as well as the surface areas and volumes of three-dimensional objects.

Heron's Area Formula

The Law of Cosines can be used to establish the following formula for the area of a triangle. This formula is called **Heron's Area Formula** after the Greek mathematician Heron (c. 100 B.C.).

Heron's Area Formula

Given any triangle with sides of lengths a, b, and c, the area of the triangle is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a + b + c)/2$.

For a proof of Heron's Area Formula, see Proofs in Mathematics on page 491.

Activities

1. Determine whether the Law of Sines or the Law of Cosines is needed to solve each of the triangles.
   a. $A = 15°, B = 58°, c = 94$
   b. $a = 96, b = 43, A = 105°$
   c. $a = 24, b = 16, c = 29$
   d. $a = 15, c = 42, B = 49°$

   Answer: a. Law of Sines, b. Law of Sines, c. Law of Cosines, d. Law of Cosines
2. Solve the triangle: $a = 31, b = 52, c = 28$.

   Answer: $A = 29.8°, B = 123.5°, C = 26.7°$
3. Use Heron's Area Formula to find the area of a triangle with sides of lengths $a = 31, b = 52$, and $c = 28$.

   Answer: Area $\approx$ 361.8 square units

Example 5 Using Heron's Area Formula

Find the area of a triangle having sides of lengths $a = 43$ meters, $b = 53$ meters, and $c = 72$ meters.

Solution

Because $s = (a + b + c)/2 = 168/2 = 84$, Heron's Area Formula yields

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{84(41)(31)(12)} \approx 1131.89 \text{ square meters.}$$

CHECKPOINT Now try Exercise 47.

You have now studied three different formulas for the area of a triangle.

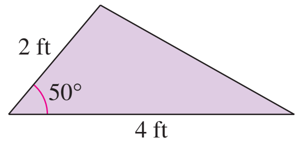
Standard Formula Area $= \frac{1}{2}bh$

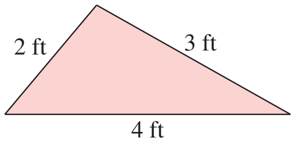
Oblique Triangle Area $= \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B$

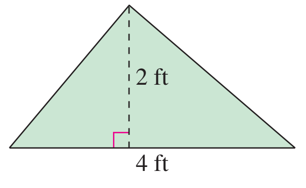
Heron's Area Formula Area $= \sqrt{s(s-a)(s-b)(s-c)}$

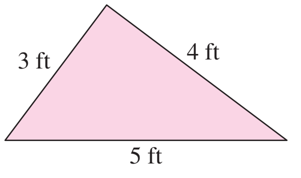
Writing About Mathematics

The Area of a Triangle Use the most appropriate formula to find the area of each triangle below. Show your work and give your reasons for choosing each formula.

a.


b.


c.


d.


Writing About Mathematics Suggestion:

You may want to ask students to draw the diagram of a triangle (labeling whatever sides and/or angles are necessary) that is well-suited to each area formula. Then ask students to exchange triangles with a partner to find the area of each.

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6.2 Exercises

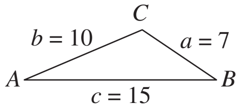
VOCABULARY CHECK: Fill in the blanks.

1. If you are given three sides of a triangle, you would use the Law of ________ to find the three angles of the triangle.
2. The standard form of the Law of Cosines for $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$ is ________ .
3. The Law of Cosines can be used to establish a formula for finding the area of a triangle called ________ ________ Formula.

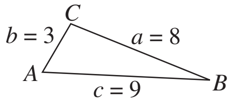
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–16, use the Law of Cosines to solve the triangle. Round your answers to two decimal places.

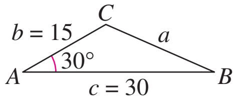
1.



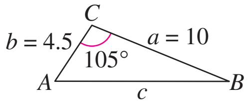
2.



3.



4.



5. $a = 11, \quad b = 14, \quad c = 20$

6. $a = 55, \quad b = 25, \quad c = 72$

7. $a = 75.4, \quad b = 52, \quad c = 52$

8. $a = 1.42, \quad b = 0.75, \quad c = 1.25$

9. $A = 135°, \quad b = 4, \quad c = 9$

10. $A = 55°, \quad b = 3, \quad c = 10$

11. $B = 10° 35', \quad a = 40, \quad c = 30$

12. $B = 75° 20', \quad a = 6.2, \quad c = 9.5$

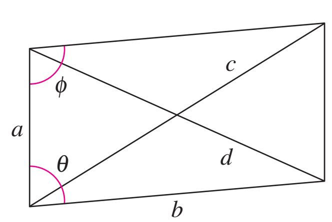
13. $B = 125° 40', \quad a = 32, \quad c = 32$

14. $C = 15° 15', \quad a = 6.25, \quad b = 2.15$

15. $C = 43°, \quad a = \frac{4}{9}, \quad b = \frac{7}{9}$

16. $C = 103°, \quad a = \frac{3}{8}, \quad b = \frac{3}{4}$

In Exercises 17–22, complete the table by solving the parallelogram shown in the figure. (The lengths of the diagonals are given by *c* and *d*.)



	a	b	c	d	θ	ϕ
17.	5	8			45°	
18.	25	35				120°
19.	10	14	20			
20.	40	60		80		
21.	15		25	20		
22.		25	50	35		

In Exercises 23–28, use Heron's Area Formula to find the area of the triangle.

23. $a = 5, \quad b = 7, \quad c = 10$

24. $a = 12, \quad b = 15, \quad c = 9$

25. $a = 2.5, \quad b = 10.2, \quad c = 9$

26. $a = 75.4, \quad b = 52, \quad c = 52$

27. $a = 12.32, \quad b = 8.46, \quad c = 15.05$

28. $a = 3.05, \quad b = 0.75, \quad c = 2.45$

29. ***Navigation*** A boat race runs along a triangular course marked by buoys A, B, and C. The race starts with the boats headed west for 3700 meters. The other two sides of the course lie to the north of the first side, and their lengths are 1700 meters and 3000 meters. Draw a figure that gives a visual representation of the problem, and find the bearings for the last two legs of the race.

30. ***Navigation*** A plane flies 810 miles from Franklin to Centerville with a bearing of 75°. Then it flies 648 miles from Centerville to Rosemount with a bearing of 32°. Draw a figure that visually represents the problem, and find the straight-line distance and bearing from Franklin to Rosemount.

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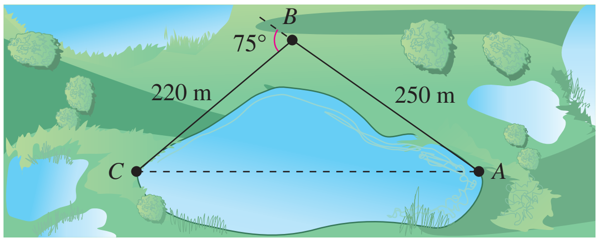
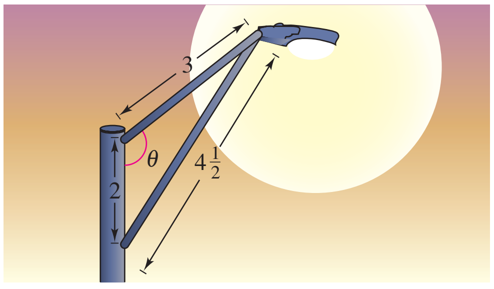
31. ***Surveying*** To approximate the length of a marsh, a surveyor walks 250 meters from point A to point B, then turns 75° and walks 220 meters to point C (see figure). Approximate the length AC of the marsh.



32. ***Surveying*** A triangular parcel of land has 115 meters of frontage, and the other boundaries have lengths of 76 meters and 92 meters. What angles does the frontage make with the two other boundaries?

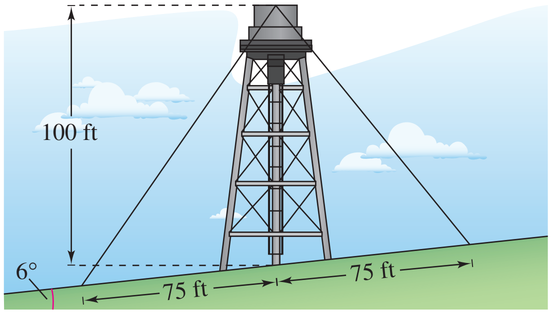
33. ***Surveying*** A triangular parcel of ground has sides of lengths 725 feet, 650 feet, and 575 feet. Find the measure of the largest angle.

34. ***Streetlight Design*** Determine the angle θ in the design of the streetlight shown in the figure.



35. ***Distance*** Two ships leave a port at 9 A.M. One travels at a bearing of N 53° W at 12 miles per hour, and the other travels at a bearing of S 67° W at 16 miles per hour. Approximate how far apart they are at noon that day.

36. ***Length*** A 100-foot vertical tower is to be erected on the side of a hill that makes a 6° angle with the horizontal (see figure). Find the length of each of the two guy wires that will be anchored 75 feet uphill and downhill from the base of the tower.

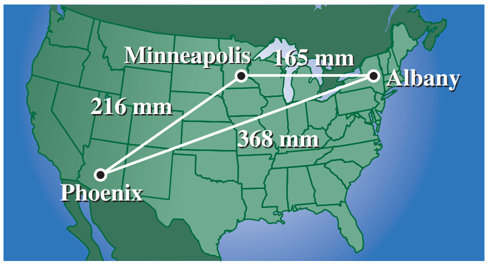


37. ***Navigation*** On a map, Orlando is 178 millimeters due south of Niagara Falls, Denver is 273 millimeters from Orlando, and Denver is 235 millimeters from Niagara Falls (see figure).

(a) Find the bearing of Denver from Orlando.

(b) Find the bearing of Denver from Niagara Falls.

38. ***Navigation*** On a map, Minneapolis is 165 millimeters due west of Albany, Phoenix is 216 millimeters from Minneapolis, and Phoenix is 368 millimeters from Albany (see figure).

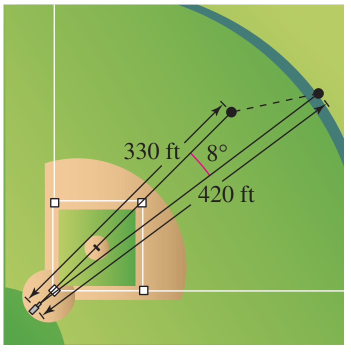


(a) Find the bearing of Minneapolis from Phoenix.

(b) Find the bearing of Albany from Phoenix.

39. ***Baseball*** On a baseball diamond with 90-foot sides, the pitcher's mound is 60.5 feet from home plate. How far is it from the pitcher's mound to third base?

40. ***Baseball*** The baseball player in center field is playing approximately 330 feet from the television camera that is behind home plate. A batter hits a fly ball that goes to the wall 420 feet from the camera (see figure). The camera turns 8° to follow the play. Approximately how far does the center fielder have to run to make the catch?



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41. ***Aircraft Tracking*** To determine the distance between two aircraft, a tracking station continuously determines the distance to each aircraft and the angle A between them (see figure). Determine the distance a between the planes when $A = 42°$, $b = 35$ miles, and $c = 20$ miles.

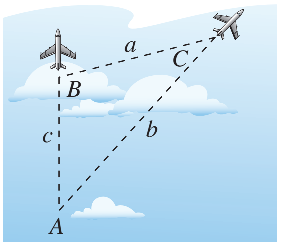
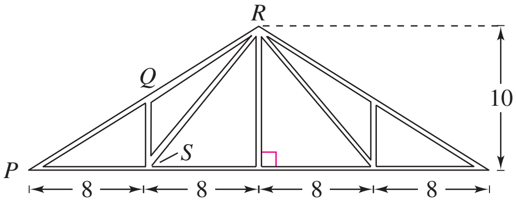


FIGURE FOR 41

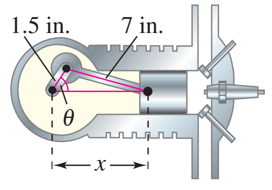
42. ***Aircraft Tracking*** Use the figure for Exercise 41 to determine the distance a between the planes when $A = 11°$, $b = 20$ miles, and $c = 20$ miles.

43. ***Trusses*** Q is the midpoint of the line segment $\overline{PR}$ in the truss rafter shown in the figure. What are the lengths of the line segments $\overline{PQ}$, $\overline{QS}$, and $\overline{RS}$?



Model It

44. ***Engine Design*** An engine has a seven-inch connecting rod fastened to a crank (see figure).



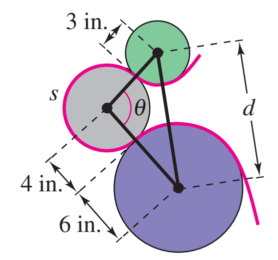
(a) Use the Law of Cosines to write an equation giving the relationship between x and θ.

(b) Write x as a function of θ. (Select the sign that yields positive values of x.)

(c) Use a graphing utility to graph the function in part (b).

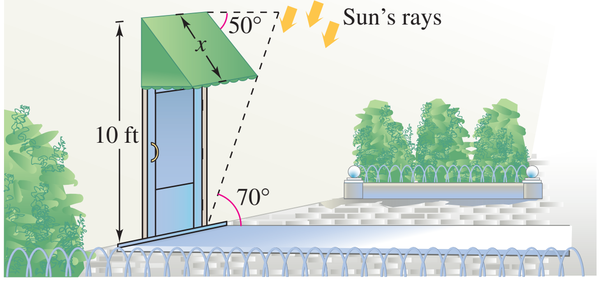
(d) Use the graph in part (c) to determine the maximum distance the piston moves in one cycle.

45. ***Paper Manufacturing*** In a process with continuous paper, the paper passes across three rollers of radii 3 inches, 4 inches, and 6 inches (see figure). The centers of the three-inch and six-inch rollers are d inches apart, and the length of the arc in contact with the paper on the four-inch roller is s inches. Complete the table.



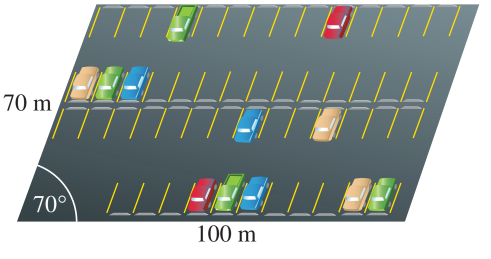
d (inches)	9	10	12	13	14	15	16
θ (degrees)							
s (inches)							

46. ***Awning Design*** A retractable awning above a patio door lowers at an angle of 50° from the exterior wall at a height of 10 feet above the ground (see figure). No direct sunlight is to enter the door when the angle of elevation of the sun is greater than 70°. What is the length x of the awning?



47. ***Geometry*** The lengths of the sides of a triangular parcel of land are approximately 200 feet, 500 feet, and 600 feet. Approximate the area of the parcel.

48. ***Geometry*** A parking lot has the shape of a parallelogram (see figure). The lengths of two adjacent sides are 70 meters and 100 meters. The angle between the two sides is 70°. What is the area of the parking lot?



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49. ***Geometry*** You want to buy a triangular lot measuring 510 yards by 840 yards by 1120 yards. The price of the land is \$2000 per acre. How much does the land cost? (*Hint:* 1 acre $=$ 4840 square yards)

50. ***Geometry*** You want to buy a triangular lot measuring 1350 feet by 1860 feet by 2490 feet. The price of the land is \$2200 per acre. How much does the land cost? (*Hint:* 1 acre $=$ 43,560 square feet)

Synthesis

True or False? **In Exercises 51–53, determine whether the statement is true or false. Justify your answer.**

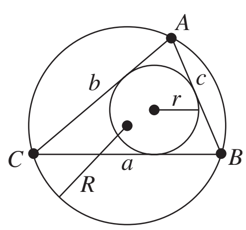
51. In Heron's Area Formula, s is the average of the lengths of the three sides of the triangle.

52. In addition to SSS and SAS, the Law of Cosines can be used to solve triangles with SSA conditions.

53. A triangle with side lengths of 10 centimeters, 16 centimeters, and 5 centimeters can be solved using the Law of Cosines.

54. ***Circumscribed and Inscribed Circles*** Let R and r be the radii of the circumscribed and inscribed circles of a triangle ABC, respectively (see figure), and let

$$s = \frac{a+b+c}{2}.$$



(a) Prove that $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

(b) Prove that $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$.

Circumscribed and Inscribed Circles **In Exercises 55 and 56, use the results of Exercise 54.**

55. Given a triangle with

$a = 25$, $b = 55$, and $c = 72$

find the areas of (a) the triangle, (b) the circumscribed circle, and (c) the inscribed circle.

56. Find the length of the largest circular running track that can be built on a triangular piece of property with sides of lengths 200 feet, 250 feet, and 325 feet.

57. ***Proof*** Use the Law of Cosines to prove that

$$\frac{1}{2}bc(1+\cos A) = \frac{a+b+c}{2}\cdot\frac{-a+b+c}{2}.$$

58. ***Proof*** Use the Law of Cosines to prove that

$$\frac{1}{2}bc(1-\cos A) = \frac{a-b+c}{2}\cdot\frac{a+b-c}{2}.$$

Skills Review

In Exercises 59–64, evaluate the expression without using a calculator.

59. $\arcsin(-1)$

60. $\arccos 0$

61. $\arctan \sqrt{3}$

62. $\arctan(-\sqrt{3})$

63. $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$

64. $\arccos\left(-\dfrac{\sqrt{3}}{2}\right)$

In Exercises 65–68, write an algebraic expression that is equivalent to the expression.

65. $\sec(\arcsin 2x)$

66. $\tan(\arccos 3x)$

67. $\cot[\arctan(x-2)]$

68. $\cos\left(\arcsin \dfrac{x-1}{2}\right)$

In Exercises 69–72, use trigonometric substitution to write the algebraic equation as a trigonometric function of θ, where $-\pi/2 < \theta < \pi/2$. Then find $\sec\theta$ and $\csc\theta$.

69. $5 = \sqrt{25-x^2}, \quad x = 5\sin\theta$

70. $-\sqrt{2} = \sqrt{4-x^2}, \quad x = 2\cos\theta$

71. $-\sqrt{3} = \sqrt{x^2-9}, \quad x = 3\sec\theta$

72. $12 = \sqrt{36+x^2}, \quad x = 6\tan\theta$

In Exercises 73 and 74, write the sum or difference as a product.

73. $\cos\dfrac{5\pi}{6} - \cos\dfrac{\pi}{3}$

74. $\sin\left(x-\dfrac{\pi}{2}\right) - \sin\left(x+\dfrac{\pi}{2}\right)$