### 4.3 Right Triangle Trigonometry

## What you should learn

- Evaluate trigonometric functions of acute angles.
- Use the fundamental trigonometric identities.
- Use a calculator to evaluate trigonometric functions.
- Use trigonometric functions to model and solve real-life problems.


## Why you should learn it

Trigonometric functions are often used to analyze real-life situations. For instance, in Exercise 71 on page 311 , you can use trigonometric functions to find the height of a helium-filled balloon.


## The Six Trigonometric Functions

Our second look at the trigonometric functions is from a right triangle perspective. Consider a right triangle, with one acute angle labeled $\theta$, as shown in Figure 4.26. Relative to the angle $\theta$, the three sides of the triangle are the hypotenuse, the opposite side (the side opposite the angle $\theta$ ), and the adjacent side (the side adjacent to the angle $\theta$ ).


Using the lengths of these three sides, you can form six ratios that define the six trigonometric functions of the acute angle $\theta$.
sine cosecant cosine secant tangent cotangent
In the following definitions, it is important to see that $0^{\circ}<\theta<90^{\circ}(\theta$ lies in the first quadrant) and that for such angles the value of each trigonometric function is positive.

## Right Triangle Definitions of Trigonometric Functions

Let $\theta$ be an acute angle of a right triangle. The six trigonometric functions of the angle $\theta$ are defined as follows. (Note that the functions in the second row are the reciprocals of the corresponding functions in the first row.)

$$
\begin{array}{lll}
\sin \theta=\frac{\text { opp }}{\text { hyp }} & \cos \theta=\frac{\text { adj }}{\text { hyp }} & \tan \theta=\frac{\text { opp }}{\text { adj }} \\
\csc \theta=\frac{\text { hyp }}{\text { opp }} & \sec \theta=\frac{\text { hyp }}{\text { adj }} & \cot \theta=\frac{\text { adj }}{\text { opp }}
\end{array}
$$

The abbreviations opp, adj, and hyp represent the lengths of the three sides of a right triangle.
opp $=$ the length of the side opposite $\theta$
adj $=$ the length of the side adjacent to $\theta$
hyp $=$ the length of the hypotenuse


FIGURE 4.27

You may wish to review the Pythagorean Theorem before presenting the examples in this section.

## Historical Note

Georg Joachim Rhaeticus (1514-1576) was the leading Teutonic mathematical astronomer of the 16th century. He was the first to define the trigonometric functions as ratios of the sides of a right triangle.


FIGURE 4.28

## Example 1 Evaluating Trigonometric Functions

Use the triangle in Figure 4.27 to find the values of the six trigonometric functions of $\theta$.

## Solution

By the Pythagorean Theorem, $(\mathrm{hyp})^{2}=(\mathrm{opp})^{2}+(\mathrm{adj})^{2}$, it follows that

$$
\begin{aligned}
\text { hyp } & =\sqrt{4^{2}+3^{2}} \\
& =\sqrt{25} \\
& =5 .
\end{aligned}
$$

So, the six trigonometric functions of $\theta$ are

$$
\begin{array}{ll}
\sin \theta=\frac{\text { opp }}{\text { hyp }}=\frac{4}{5} & \csc \theta=\frac{\text { hyp }}{\text { opp }}=\frac{5}{4} \\
\cos \theta=\frac{\text { adj }}{\text { hyp }}=\frac{3}{5} & \sec \theta=\frac{\text { hyp }}{\text { adj }}=\frac{5}{3} \\
\tan \theta=\frac{\text { opp }}{\text { adj }}=\frac{4}{3} & \cot \theta=\frac{\text { adj }}{\text { opp }}=\frac{3}{4} .
\end{array}
$$

## CHECKPOINT Now try Exercise 3.

In Example 1, you were given the lengths of two sides of the right triangle, but not the angle $\theta$. Often, you will be asked to find the trigonometric functions of a given acute angle $\theta$. To do this, construct a right triangle having $\theta$ as one of its angles.

## Example 2 Evaluating Trigonometric Functions of $45^{\circ}$

Find the values of $\sin 45^{\circ}, \cos 45^{\circ}$, and $\tan 45^{\circ}$.

## Solution

Construct a right triangle having $45^{\circ}$ as one of its acute angles, as shown in Figure 4.28. Choose the length of the adjacent side to be 1. From geometry, you know that the other acute angle is also $45^{\circ}$. So, the triangle is isosceles and the length of the opposite side is also 1 . Using the Pythagorean Theorem, you find the length of the hypotenuse to be $\sqrt{2}$.

$$
\begin{aligned}
& \sin 45^{\circ}=\frac{\text { opp }}{\text { hyp }}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2} \\
& \cos 45^{\circ}=\frac{\text { adj }}{\text { hyp }}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}
\end{aligned}
$$

$$
\tan 45^{\circ}=\frac{\mathrm{opp}}{\mathrm{adj}}=\frac{1}{1}=1
$$

dCHECKPOINT Now try Exercise 17.

## STUDY TIP

Because the angles $30^{\circ}, 45^{\circ}$, and $60^{\circ}(\pi / 6, \pi / 4$, and $\pi / 3)$ occur frequently in trigonometry, you should learn to construct the triangles shown in Figures 4.28 and 4.29.

Consider having your students construct the triangle in Figure 4.29 with angles in the corresponding radian measures, then find the six trigonometric functions for each of the acute angles.

## Technology

You can use a calculator to convert the answers in Example 3 to decimals. However, the radical form is the exact value and in most cases, the exact value is preferred.

The triangles in Figures 4.27, 4.28, and 4.29 are useful problem-solving aids. Encourage your students to draw diagrams when they solve problems similar to those in Examples 1, 2, and 3.

## Example 3 Evaluating Trigonometric Functions of $30^{\circ}$ and $60^{\circ}$

Use the equilateral triangle shown in Figure 4.29 to find the values of $\sin 60^{\circ}$, $\cos 60^{\circ}, \sin 30^{\circ}$, and $\cos 30^{\circ}$.


FIGURE 4.29

## Solution

Use the Pythagorean Theorem and the equilateral triangle in Figure 4.29 to verify the lengths of the sides shown in the figure. For $\theta=60^{\circ}$, you have adj $=1$, opp $=\sqrt{3}$, and hyp $=2$. So,

$$
\sin 60^{\circ}=\frac{\text { opp }}{\text { hyp }}=\frac{\sqrt{3}}{2} \quad \text { and } \quad \cos 60^{\circ}=\frac{\text { adj }}{\text { hyp }}=\frac{1}{2} .
$$

For $\theta=30^{\circ}$, adj $=\sqrt{3}$, opp $=1$, and hyp $=2$. So,

$$
\sin 30^{\circ}=\frac{\text { opp }}{\text { hyp }}=\frac{1}{2} \quad \text { and } \quad \cos 30^{\circ}=\frac{\text { adj }}{\text { hyp }}=\frac{\sqrt{3}}{2} .
$$

## dCHECKPOINT Now try Exercise 19.

## Sines, Cosines, and Tangents of Special Angles

$\sin 30^{\circ}=\sin \frac{\pi}{6}=\frac{1}{2} \quad \cos 30^{\circ}=\cos \frac{\pi}{6}=\frac{\sqrt{3}}{2} \quad \tan 30^{\circ}=\tan \frac{\pi}{6}=\frac{\sqrt{3}}{3}$
$\sin 45^{\circ}=\sin \frac{\pi}{4}=\frac{\sqrt{2}}{2} \quad \cos 45^{\circ}=\cos \frac{\pi}{4}=\frac{\sqrt{2}}{2} \quad \tan 45^{\circ}=\tan \frac{\pi}{4}=1$
$\sin 60^{\circ}=\sin \frac{\pi}{3}=\frac{\sqrt{3}}{2} \quad \cos 60^{\circ}=\cos \frac{\pi}{3}=\frac{1}{2} \quad \tan 60^{\circ}=\tan \frac{\pi}{3}=\sqrt{3}$

In the box, note that $\sin 30^{\circ}=\frac{1}{2}=\cos 60^{\circ}$. This occurs because $30^{\circ}$ and $60^{\circ}$ are complementary angles. In general, it can be shown from the right triangle definitions that cofunctions of complementary angles are equal. That is, if $\theta$ is an acute angle, the following relationships are true.

$$
\begin{array}{ll}
\sin \left(90^{\circ}-\theta\right)=\cos \theta & \cos \left(90^{\circ}-\theta\right)=\sin \theta \\
\tan \left(90^{\circ}-\theta\right)=\cot \theta & \cot \left(90^{\circ}-\theta\right)=\tan \theta \\
\sec \left(90^{\circ}-\theta\right)=\csc \theta & \csc \left(90^{\circ}-\theta\right)=\sec \theta
\end{array}
$$

These identities will be used many times in trigonometry and later in calculus. Encourage your students to learn them well.


FIGURE 4.30

## Trigonometric Identities

In trigonometry, a great deal of time is spent studying relationships between trigonometric functions (identities).

## Fundamental Trigonometric Identities

Reciprocal Identities

$$
\begin{array}{lll}
\sin \theta=\frac{1}{\csc \theta} & \cos \theta=\frac{1}{\sec \theta} & \tan \theta=\frac{1}{\cot \theta} \\
\csc \theta=\frac{1}{\sin \theta} & \sec \theta=\frac{1}{\cos \theta} & \cot \theta=\frac{1}{\tan \theta}
\end{array}
$$

Quotient Identities

$$
\tan \theta=\frac{\sin \theta}{\cos \theta} \quad \cot \theta=\frac{\cos \theta}{\sin \theta}
$$

Pythagorean Identities

$$
\begin{array}{rl}
\sin ^{2} \theta+\cos ^{2} \theta=1 & 1+\tan ^{2} \theta \\
1+\cot ^{2} \theta & =\sec ^{2} \theta \\
& 1+
\end{array}
$$

Note that $\sin ^{2} \theta$ represents $(\sin \theta)^{2}, \cos ^{2} \theta$ represents $(\cos \theta)^{2}$, and so on.

## Example 4 Applying Trigonometric Identities

Let $\theta$ be an acute angle such that $\sin \theta=0.6$. Find the values of (a) $\cos \theta$ and (b) $\tan \theta$ using trigonometric identities.

## Solution

a. To find the value of $\cos \theta$, use the Pythagorean identity

$$
\sin ^{2} \theta+\cos ^{2} \theta=1
$$

So, you have

$$
\begin{aligned}
(0.6)^{2}+\cos ^{2} \theta & =1 & & \text { Substitute } 0.6 \text { for } \sin \theta . \\
\cos ^{2} \theta & =1-(0.6)^{2}=0.64 & & \text { Subtract }(0.6)^{2} \text { from each side. } \\
\cos \theta & =\sqrt{0.64}=0.8 . & & \text { Extract the positive square root. }
\end{aligned}
$$

b. Now, knowing the sine and cosine of $\theta$, you can find the tangent of $\theta$ to be

$$
\begin{aligned}
\tan \theta & =\frac{\sin \theta}{\cos \theta} \\
& =\frac{0.6}{0.8} \\
& =0.75 .
\end{aligned}
$$

Use the definitions of $\cos \theta$ and $\tan \theta$, and the triangle shown in Figure 4.30, to check these results.
(Checkpont Now try Exercise 29.


FIGURE 4.31

## STUDY TIP

You can also use the reciprocal identities for sine, cosine, and tangent to evaluate the cosecant, secant, and cotangent functions with a calculator. For instance, you could use the following keystroke sequence to evaluate $\sec 28^{\circ}$.

## $1 \doteqdot$ COS 28 ENTER

The calculator should display 1.1325701 .

One of the most common errors students make when they evaluate trigonometric functions with a calculator is not having their calculators set to the correct mode (radian vs. degree).

## Example 5 Applying Trigonometric Identities

Let $\theta$ be an acute angle such that $\tan \theta=3$. Find the values of (a) $\cot \theta$ and (b) $\sec \theta$ using trigonometric identities.

## Solution

a. $\cot \theta=\frac{1}{\tan \theta}$

Reciprocal identity
$\cot \theta=\frac{1}{3}$
b. $\sec ^{2} \theta=1+\tan ^{2} \theta \quad$ Pythagorean identity

$$
\begin{aligned}
\sec ^{2} \theta & =1+3^{2} \\
\sec ^{2} \theta & =10 \\
\sec \theta & =\sqrt{10}
\end{aligned}
$$

Use the definitions of $\cot \theta$ and $\sec \theta$, and the triangle shown in Figure 4.31, to check these results.
$\checkmark$ CHECKPOINT Now try Exercise 31.

## Evaluating Trigonometric Functions with a Calculator

To use a calculator to evaluate trigonometric functions of angles measured in degrees, first set the calculator to degree mode and then proceed as demonstrated in Section 4.2. For instance, you can find values of $\cos 28^{\circ}$ and $\sec 28^{\circ}$ as follows.

|  | Function | Mode | Calculator Keystrokes |
| :--- | :--- | :--- | :---: |
| a. $\cos 28^{\circ}$ | Degree | COS 28 ENTER | 0.8829476 |
| b. $\sec 28^{\circ}$ | Degree | $\square \operatorname{COS} \square 28 \square \square x^{-1}$ ENTER | 1.1325701 |

Throughout this text, angles are assumed to be measured in radians unless noted otherwise. For example, $\sin 1$ means the sine of 1 radian and $\sin 1^{\circ}$ means the sine of 1 degree.

## Example 6 Using a Calculator

Use a calculator to evaluate $\sec \left(5^{\circ} 40^{\prime} 12^{\prime \prime}\right)$.

## Solution

Begin by converting to decimal degree form. [Recall that $1^{\prime}=\frac{1}{60}\left(1^{\circ}\right)$ and $\left.1^{\prime \prime}=\frac{1}{3600}\left(1^{\circ}\right)\right]$.

$$
5^{\circ} 40^{\prime} 12^{\prime \prime}=5^{\circ}+\left(\frac{40}{60}\right)^{\circ}+\left(\frac{12}{3600}\right)^{\circ}=5.67^{\circ}
$$

Then, use a calculator to evaluate sec $5.67^{\circ}$.
Function
Calculator Keystrokes
Display
$\sec \left(5^{\circ} 40^{\prime} 12^{\prime \prime}\right)=\sec 5.67^{\circ} \square \operatorname{COS} 0.67 \square \square x^{-1}$ ENTER 1.0049166
(CHECKPOINT Now try Exercise 47.


FIGURE 4.32


FIGURE 4.33

## Applications Involving Right Triangles

Many applications of trigonometry involve a process called solving right triangles. In this type of application, you are usually given one side of a right triangle and one of the acute angles and are asked to find one of the other sides, or you are given two sides and are asked to find one of the acute angles.

In Example 7, the angle you are given is the angle of elevation, which represents the angle from the horizontal upward to an object. For objects that lie below the horizontal, it is common to use the term angle of depression, as shown in Figure 4.32.

## Example 7 Using Trigonometry to Solve a Right Triangle

A surveyor is standing 115 feet from the base of the Washington Monument, as shown in Figure 4.33. The surveyor measures the angle of elevation to the top of the monument as $78.3^{\circ}$. How tall is the Washington Monument?

## Solution

From Figure 4.33, you can see that

$$
\tan 78.3^{\circ}=\frac{\mathrm{opp}}{\mathrm{adj}}=\frac{y}{x}
$$

where $x=115$ and $y$ is the height of the monument. So, the height of the Washington Monument is

$$
y=x \tan 78.3^{\circ} \approx 115(4.82882) \approx 555 \text { feet. }
$$

(CHECKPOINT Now try Exercise 63.

## Example 8 Using Trigonometry to Solve a Right Triangle

An historic lighthouse is 200 yards from a bike path along the edge of a lake. A walkway to the lighthouse is 400 yards long. Find the acute angle $\theta$ between the bike path and the walkway, as illustrated in Figure 4.34.


FIGURE 4.34

## Solution

From Figure 4.34, you can see that the sine of the angle $\theta$ is

$$
\sin \theta=\frac{\text { opp }}{\text { hyp }}=\frac{200}{400}=\frac{1}{2} .
$$

Now you should recognize that $\theta=30^{\circ}$.
dCHECKPOINT Now try Exercise 65.

By now you are able to recognize that $\theta=30^{\circ}$ is the acute angle that satisfies the equation $\sin \theta=\frac{1}{2}$. Suppose, however, that you were given the equation $\sin \theta=0.6$ and were asked to find the acute angle $\theta$. Because

$$
\begin{aligned}
\sin 30^{\circ} & =\frac{1}{2} \\
& =0.5000
\end{aligned}
$$

and

$$
\begin{aligned}
\sin 45^{\circ} & =\frac{1}{\sqrt{2}} \\
& \approx 0.7071
\end{aligned}
$$

you might guess that $\theta$ lies somewhere between $30^{\circ}$ and $45^{\circ}$. In a later section, you will study a method by which a more precise value of $\theta$ can be determined.

## Example 9 Solving a Right Triangle

Find the length $c$ of the skateboard ramp shown in Figure 4.35.


FIGURE 4.35

## Solution

From Figure 4.35, you can see that

$$
\begin{aligned}
\sin 18.4^{\circ} & =\frac{\mathrm{opp}}{\mathrm{hyp}} \\
& =\frac{4}{c} .
\end{aligned}
$$

So, the length of the skateboard ramp is

$$
\begin{aligned}
c & =\frac{4}{\sin 18.4^{\circ}} \\
& \approx \frac{4}{0.3156} \\
& \approx 12.7 \text { feet. }
\end{aligned}
$$

(CHECKPOINT Now try Exercise 67.

### 4.3 Exercises

## VOCABULARY CHECK:

1. Match the trigonometric function with its right triangle definition.
(a) Sine
(b) Cosine
(c) Tangent
(d) Cosecant
(e) Secant
(f) Cotangent
(i) $\frac{\text { hypotenuse }}{\text { adjacent }}$
(ii) $\frac{\text { adjacent }}{\text { opposite }}$
(iii) $\frac{\text { hypotenuse }}{\text { opposite }}$
(iv) $\frac{\text { adjacent }}{\text { hypotenuse }}$
(v) $\frac{\text { opposite }}{\text { hypotenuse }}$
(vi) $\frac{\text { opposite }}{\text { adjacent }}$

In Exercises 2 and 3, fill in the blanks.
2. Relative to the angle $\theta$, the three sides of a right triangle are the $\qquad$ side, the $\qquad$ side, and the $\qquad$ _.
3. An angle that measures from the horizontal upward to an object is called the angle of $\qquad$ whereas an angle that measures from the horizontal downward to an object is called the angle of $\qquad$ .

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1-4, find the exact values of the six trigonometric functions of the angle $\theta$ shown in the figure. (Use the Pythagorean Theorem to find the third side of the triangle.)
1.

2.

3.

4.


In Exercises 5-8, find the exact values of the six trigonometric functions of the angle $\theta$ for each of the two triangles. Explain why the function values are the same.
5.

6.


7.

8.


In Exercises 9-16, sketch a right triangle corresponding to the trigonometric function of the acute angle $\boldsymbol{\theta}$. Use the Pythagorean Theorem to determine the third side and then find the other five trigonometric functions of $\boldsymbol{\theta}$.
9. $\sin \theta=\frac{3}{4}$
10. $\cos \theta=\frac{5}{7}$
11. $\sec \theta=2$
12. $\cot \theta=5$
13. $\tan \theta=3$
14. $\sec \theta=6$
15. $\cot \theta=\frac{3}{2}$
16. $\csc \theta=\frac{17}{4}$

In Exercises 17-26, construct an appropriate triangle to complete the table. $\left(0 \leq \theta \leq 90^{\circ}, 0 \leq \theta \leq \pi / 2\right)$

| Function | $\theta(\mathrm{deg})$ | $\theta(\mathrm{rad})$ | Function Value |
| :---: | :---: | :---: | :---: |
| 17. $\sin$ | $30^{\circ}$ |  |  |
| 18. $\cos$ | $45^{\circ}$ |  |  |
| 19. $\tan$ |  | $\frac{\pi}{3}$ |  |
| 20. sec |  | $\frac{\pi}{4}$ |  |
| 21. $\cot$ |  |  | $\frac{\sqrt{3}}{3}$ |
| 22. csc |  |  | $\sqrt{2}$ |
| 23. $\cos$ |  | $\frac{\pi}{6}$ |  |
| 24. $\sin$ |  | $\frac{\pi}{4}$ |  |
| 25. $\cot$ |  |  | 1 |
| 26. $\tan$ |  |  | $\frac{\sqrt{3}}{3}$ |

In Exercises 27-32, use the given function value(s), and trigonometric identities (including the cofunction identities), to find the indicated trigonometric functions.
27. $\sin 60^{\circ}=\frac{\sqrt{3}}{2}, \quad \cos 60^{\circ}=\frac{1}{2}$
(a) $\tan 60^{\circ}$
(b) $\sin 30^{\circ}$
(c) $\cos 30^{\circ}$
(d) $\cot 60^{\circ}$
28. $\sin 30^{\circ}=\frac{1}{2}, \quad \tan 30^{\circ}=\frac{\sqrt{3}}{3}$
(a) $\csc 30^{\circ}$
(b) $\cot 60^{\circ}$
(c) $\cos 30^{\circ}$
(d) $\cot 30^{\circ}$
29. $\csc \theta=\frac{\sqrt{13}}{2}, \quad \sec \theta=\frac{\sqrt{13}}{3}$
(a) $\sin \theta$
(b) $\cos \theta$
(c) $\tan \theta$
(d) $\sec \left(90^{\circ}-\theta\right)$
30. $\sec \theta=5, \quad \tan \theta=2 \sqrt{6}$
(a) $\cos \theta$
(b) $\cot \theta$
(c) $\cot \left(90^{\circ}-\theta\right)$
(d) $\sin \theta$
31. $\cos \alpha=\frac{1}{3}$
(a) $\sec \alpha$
(b) $\sin \alpha$
(c) $\cot \alpha$
(d) $\sin \left(90^{\circ}-\alpha\right)$
32. $\tan \beta=5$
(a) $\cot \beta$
(b) $\cos \beta$
(c) $\tan \left(90^{\circ}-\beta\right)$
(d) $\csc \beta$

In Exercises 33-42, use trigonometric identities to transform the left side of the equation into the right side ( $0<\theta<\pi / 2$ ).
33. $\tan \theta \cot \theta=1$
34. $\cos \theta \sec \theta=1$
35. $\tan \alpha \cos \alpha=\sin \alpha$
36. $\cot \alpha \sin \alpha=\cos \alpha$
37. $(1+\cos \theta)(1-\cos \theta)=\sin ^{2} \theta$
38. $(1+\sin \theta)(1-\sin \theta)=\cos ^{2} \theta$
39. $(\sec \theta+\tan \theta)(\sec \theta-\tan \theta)=1$
40. $\sin ^{2} \theta-\cos ^{2} \theta=2 \sin ^{2} \theta-1$
41. $\frac{\sin \theta}{\cos \theta}+\frac{\cos \theta}{\sin \theta}=\csc \theta \sec \theta$
42. $\frac{\tan \beta+\cot \beta}{\tan \beta}=\csc ^{2} \beta$

In Exercises 43-52, use a calculator to evaluate each function. Round your answers to four decimal places. (Be sure the calculator is in the correct angle mode.)
43. (a) $\sin 10^{\circ}$
(b) $\cos 80^{\circ}$
44. (a) $\tan 23.5^{\circ}$
(b) $\cot 66.5^{\circ}$
45. (a) $\sin 16.35^{\circ}$
(b) $\csc 16.35^{\circ}$
46. (a) $\cos 16^{\circ} 18^{\prime}$
(b) $\sin 73^{\circ} 56^{\prime}$
47. (a) $\sec 42^{\circ} 12^{\prime}$
(b) $\csc 48^{\circ} 7^{\prime}$
48. (a) $\cos 4^{\circ} 50^{\prime} 15^{\prime \prime}$
(b) $\sec 4^{\circ} 50^{\prime} 15^{\prime \prime}$
49. (a) $\cot 11^{\circ} 15^{\prime}$
(b) $\tan 11^{\circ} 15^{\prime}$
50. (a) $\sec 56^{\circ} 810^{\prime \prime}$
(b) $\cos 56^{\circ} 810^{\prime \prime}$
51. (a) $\csc 32^{\circ} 40^{\prime} 3^{\prime \prime}$
(b) $\tan 44^{\circ} 2816^{\prime \prime}$
52. (a) $\sec \left(\frac{9}{5} \cdot 20+32\right)^{\circ}$
(b) $\cot \left(\frac{9}{5} \cdot 30+32\right)^{\circ}$

In Exercises 53-58, find the values of $\theta$ in degrees ( $0^{\circ}<\theta<90^{\circ}$ ) and radians $(0<\theta<\pi / 2)$ without the aid of a calculator.
53. (a) $\sin \theta=\frac{1}{2}$
(b) $\csc \theta=2$
54. (a) $\cos \theta=\frac{\sqrt{2}}{2}$
(b) $\tan \theta=1$
55. (a) $\sec \theta=2$
(b) $\cot \theta=1$
56. (a) $\tan \theta=\sqrt{3}$
(b) $\cos \theta=\frac{1}{2}$
57. (a) $\csc \theta=\frac{2 \sqrt{3}}{3}$
(b) $\sin \theta=\frac{\sqrt{2}}{2}$
58. (a) $\cot \theta=\frac{\sqrt{3}}{3}$
(b) $\sec \theta=\sqrt{2}$

In Exercises 59-62, solve for $x, y$, or $r$ as indicated.
59. Solve for $x$.

61. Solve for $x$.

60. Solve for $y$.

62. Solve for $r$.

63. Empire State Building You are standing 45 meters from the base of the Empire State Building. You estimate that the angle of elevation to the top of the 86th floor (the observatory) is $82^{\circ}$. If the total height of the building is another 123 meters above the 86th floor, what is the approximate height of the building? One of your friends is on the 86th floor. What is the distance between you and your friend?
64. Height A six-foot person walks from the base of a broadcasting tower directly toward the tip of the shadow cast by the tower. When the person is 132 feet from the tower and 3 feet from the tip of the shadow, the person's shadow starts to appear beyond the tower's shadow.
(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the tower.
(b) Use a trigonometric function to write an equation involving the unknown quantity.
(c) What is the height of the tower?
65. Angle of Elevation You are skiing down a mountain with a vertical height of 1500 feet. The distance from the top of the mountain to the base is 3000 feet. What is the angle of elevation from the base to the top of the mountain?
66. Width of a River A biologist wants to know the width $w$ of a river so in order to properly set instruments for studying the pollutants in the water. From point $A$, the biologist walks downstream 100 feet and sights to point $C$ (see figure). From this sighting, it is determined that $\theta=54^{\circ}$. How wide is the river?

67. Length A steel cable zip-line is being constructed for a competition on a reality television show. One end of the zip-line is attached to a platform on top of a 150 -foot pole. The other end of the zip-line is attached to the top of a 5 -foot stake. The angle of elevation to the platform is $23^{\circ}$ (see figure).

(a) How long is the zip-line?
(b) How far is the stake from the pole?
(c) Contestants take an average of 6 seconds to reach the ground from the top of the zip-line. At what rate are contestants moving down the line? At what rate are they dropping vertically?
68. Height of a Mountain In traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is $3.5^{\circ}$. After you drive 13 miles closer to the mountain, the angle of elevation is $9^{\circ}$. Approximate the height of the mountain.

69. Machine Shop Calculations A steel plate has the form of one-fourth of a circle with a radius of 60 centimeters. Two two-centimeter holes are to be drilled in the plate positioned as shown in the figure. Find the coordinates of the center of each hole.

70. Machine Shop Calculations A tapered shaft has a diameter of 5 centimeters at the small end and is 15 centimeters long (see figure). The taper is $3^{\circ}$. Find the diameter $d$ of the large end of the shaft.


## Model It

71. Height A 20-meter line is used to tether a heliumfilled balloon. Because of a breeze, the line makes an angle of approximately $85^{\circ}$ with the ground.
(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the balloon.
(b) Use a trigonometric function to write an equation involving the unknown quantity.
(c) What is the height of the balloon?
(d) The breeze becomes stronger and the angle the balloon makes with the ground decreases. How does this affect the triangle you drew in part (a)?
(e) Complete the table, which shows the heights (in meters) of the balloon for decreasing angle measures $\theta$.

| Angle, $\theta$ | $80^{\circ}$ | $70^{\circ}$ | $60^{\circ}$ | $50^{\circ}$ |
| :--- | :--- | :--- | :--- | :--- |
| Height |  |  |  |  |


| Angle, $\theta$ | $40^{\circ}$ | $30^{\circ}$ | $20^{\circ}$ | $10^{\circ}$ |
| :--- | :--- | :--- | :--- | :--- |
| Height |  |  |  |  |

(f) As the angle the balloon makes with the ground approaches $0^{\circ}$, how does this affect the height of the balloon? Draw a right triangle to explain your reasoning.
72. Geometry Use a compass to sketch a quarter of a circle of radius 10 centimeters. Using a protractor, construct an angle of $20^{\circ}$ in standard position (see figure). Drop a perpendicular line from the point of intersection of the terminal side of the angle and the arc of the circle. By actual measurement, calculate the coordinates $(x, y)$ of the point of intersection and use these measurements to approximate the six trigonometric functions of a $20^{\circ}$ angle.


## Synthesis

True or False? In Exercises 73-78, determine whether the statement is true or false. Justify your answer.
73. $\sin 60^{\circ} \csc 60^{\circ}=1$
74. $\sec 30^{\circ}=\csc 60^{\circ}$
75. $\sin 45^{\circ}+\cos 45^{\circ}=1$
76. $\cot ^{2} 10^{\circ}-\csc ^{2} 10^{\circ}=-1$
77. $\frac{\sin 60^{\circ}}{\sin 30^{\circ}}=\sin 2^{\circ}$
78. $\tan \left[\left(5^{\circ}\right)^{2}\right]=\tan ^{2}\left(5^{\circ}\right)$
79. Writing In right triangle trigonometry, explain why $\sin 30^{\circ}=\frac{1}{2}$ regardless of the size of the triangle.
80. Think About It You are given only the value $\tan \theta$. Is it possible to find the value of $\sec \theta$ without finding the measure of $\theta$ ? Explain.

## 81. Exploration

(a) Complete the table.

| $\theta$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\sin \theta$ |  |  |  |  |  |

(b) Is $\theta$ or $\sin \theta$ greater for $\theta$ in the interval $(0,0.5]$ ?
(c) As $\theta$ approaches 0 , how do $\theta$ and $\sin \theta$ compare? Explain.
82. Exploration
(a) Complete the table.

| $\theta$ | $0^{\circ}$ | $18^{\circ}$ | $36^{\circ}$ | $54^{\circ}$ | $72^{\circ}$ | $90^{\circ}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\sin \theta$ |  |  |  |  |  |  |
| $\cos \theta$ |  |  |  |  |  |  |

(b) Discuss the behavior of the sine function for $\theta$ in the range from $0^{\circ}$ to $90^{\circ}$.
(c) Discuss the behavior of the cosine function for $\theta$ in the range from $0^{\circ}$ to $90^{\circ}$.
(d) Use the definitions of the sine and cosine functions to explain the results of parts (b) and (c).

## Skills Review

In Exercises 83-86, perform the operations and simplify.
83. $\frac{x^{2}-6 x}{x^{2}+4 x-12} \cdot \frac{x^{2}+12 x+36}{x^{2}-36}$
84. $\frac{2 t^{2}+5 t-12}{9-4 t^{2}} \div \frac{t^{2}-16}{4 t^{2}+12 t+9}$
85. $\frac{3}{x+2}-\frac{2}{x-2}+\frac{x}{x^{2}+4 x+4}$
86. $\frac{\left(\frac{3}{x}-\frac{1}{4}\right)}{\left(\frac{12}{x}-1\right)}$

