2.5 Zeros of Polynomial Functions

What you should learn

- Use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions.
- Find rational zeros of polynomial functions.
- Find conjugate pairs of complex zeros.
- Find zeros of polynomials by factoring.
- Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.

Why you should learn it

Finding zeros of polynomial functions is an important part of solving real-life problems. For instance, in Exercise 112 on page 182, the zeros of a polynomial function can help you analyze the attendance at women's college basketball games.

The Fundamental Theorem of Algebra

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely* n zeros. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, . . . , c_n$ are complex numbers.

For a proof of the Linear Factorization Theorem, see Proofs in Mathematics on page 214.

Note that the Fundamental Theorem of Algebra and the Linear Factorization Theorem tell you only that the zeros or factors of a polynomial exist, not how to find them. Such theorems are called *existence theorems*.

Example 1 Zeros of Polynomial Functions

a. The first-degree polynomial $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$f(x) = x^2 - 6x + 9 = (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$. (This is called a *repeated zero*.)

c. The third-degree polynomial function

$$f(x) = x^3 + 4x = x(x^2 + 4) = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fourth-degree polynomial function

$$f(x) = x^4 - 1 = (x - 1)(x + 1)(x - i)(x + i)$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

CHECKPOINT Now try Exercise 1.

STUDY TIP

Recall that in order to find the zeros of a function $f(x)$, set $f(x)$ equal to 0 and solve the resulting equation for x. For instance, the function in Example 1(a) has a zero at $x = 2$ because

$$x - 2 = 0$$
$$x = 2.$$

Finding zeros of polynomial functions is a very important concept in algebra. This is a good place to discuss the fact that polynomials do not necessarily have rational zeros but may have zeros that are irrational or complex.

Fogg Art Museum

Historical Note

Although they were not contemporaries, Jean Le Rond d'Alembert (1717–1783) worked independently of Carl Gauss in trying to prove the Fundamental Theorem of Algebra. His efforts were such that, in France, the Fundamental Theorem of Algebra is frequently known as the Theorem of d'Alembert.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

The Rational Zero Test

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$ has *integer* coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

$p =$ a factor of the constant term a_0

$q =$ a factor of the leading coefficient a_n.

To use the Rational Zero Test, you should first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Having formed this list of *possible rational zeros*, use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Example 2 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of

$$f(x) = x^3 + x + 1.$$

Solution

Because the leading coefficient is 1, the possible rational zeros are ± 1, the factors of the constant term. By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1$$
$$= 3$$
$$f(-1) = (-1)^3 + (-1) + 1$$
$$= -1$$

So, you can conclude that the given polynomial has *no* rational zeros. Note from the graph of f in Figure 2.30 that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

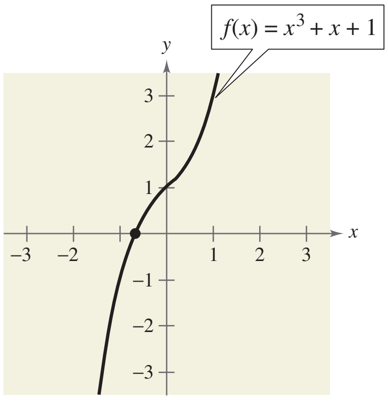


FIGURE 2.30

CHECKPOINT Now try Exercise 7.

STUDY TIP

When the list of possible rational zeros is small, as in Example 2, it may be quicker to test the zeros by evaluating the function. When the list of possible rational zeros is large, as in Example 3, it may be quicker to use a different approach to test the zeros, such as using synthetic division or sketching a graph.

Example 3 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^4 - x^3 + x^2 - 3x - 6$.

Solution

Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

By applying synthetic division successively, you can determine that $x = -1$ and $x = 2$ are the only two rational zeros.

-1	1	-1	1	-3	-6	
		-1	2	-3	6	
	1	-2	3	-6	0	0 remainder, so $x = -1$ is a zero.

2	1	-2	3	-6	
		2	0	6	
	1	0	3	0	0 remainder, so $x = 2$ is a zero.

So, $f(x)$ factors as

$$f(x) = (x + 1)(x - 2)(x^2 + 3).$$

Because the factor $(x^2 + 3)$ produces no real zeros, you can conclude that $x = -1$ and $x = 2$ are the only *real* zeros of f, which is verified in Figure 2.31.

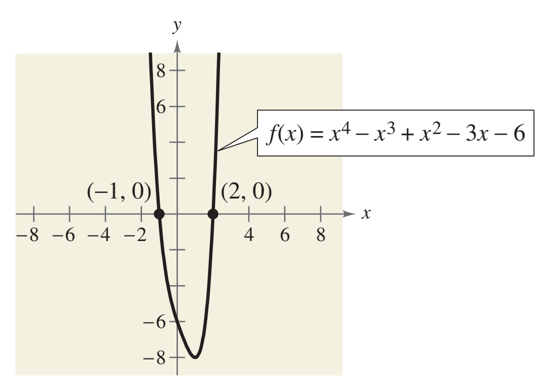


FIGURE 2.31

CHECKPOINT Now try Exercise 11.

If the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways: (1) a programmable calculator can be used to speed up the calculations; (2) a graph, drawn either by hand or with a graphing utility, can give a good estimate of the locations of the zeros; (3) the Intermediate Value Theorem along with a table generated by a graphing utility can give approximations of zeros; and (4) synthetic division can be used to test the possible rational zeros.

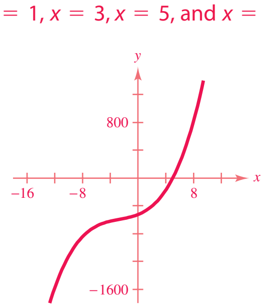
Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division, as shown in Example 3.

Additional Example

List the possible rational zeros of $f(x) = x^3 + 8x^2 + 40x - 525$.

Solution

The leading coefficient is 1, so the possible rational zeros are $\pm 1, \pm 3, \pm 5, \pm 7, \pm 15, \pm 21, \pm 25, \pm 35, \pm 75, \pm 105, \pm 175$, and ± 525. To decide which possible rational zeros should be tested using synthetic division, graph the function. From the graph, you can see that the zero is positive and less than 10, so the only values of x that should be tested are $x = 1, x = 3, x = 5$, and $x = 7$.



Example 4 Using the Rational Zero Test

Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution

The leading coefficient is 2 and the constant term is 3.

$$\textit{Possible rational zeros: } \frac{\text{Factors of 3}}{\text{Factors of 2}} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$f(x) = (x - 1)(2x^2 + 5x - 3)$$
$$= (x - 1)(2x - 1)(x + 3)$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

CHECKPOINT Now try Exercise 17.

STUDY TIP

Remember that when you try to find the rational zeros of a polynomial function with many possible rational zeros, as in Example 4, you must use trial and error. There is no quick algebraic method to determine which of the possibilities is an actual zero; however, sketching a graph may be helpful.

Recall from Section 2.2 that if $x = a$ is a zero of the polynomial function f, then $x = a$ is a solution of the polynomial equation $f(x) = 0$.

Example 5 Solving a Polynomial Equation

Find all the real solutions of $-10x^3 + 15x^2 + 16x - 12 = 0$.

Solution

The leading coefficient is -10 and the constant term is -12.

$$\textit{Possible rational solutions: } \frac{\text{Factors of } -12}{\text{Factors of } -10} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

With so many possibilities (32, in fact), it is worth your time to stop and sketch a graph. From Figure 2.32, it looks like three reasonable solutions would be $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$. Testing these by synthetic division shows that $x = 2$ is the only rational solution. So, you have

$$(x - 2)(-10x^2 - 5x + 6) = 0.$$

Using the Quadratic Formula for the second factor, you find that the two additional solutions are irrational numbers.

$$x = \frac{-5 - \sqrt{265}}{20} \approx -1.0639$$

and

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.5639$$

CHECKPOINT Now try Exercise 23.

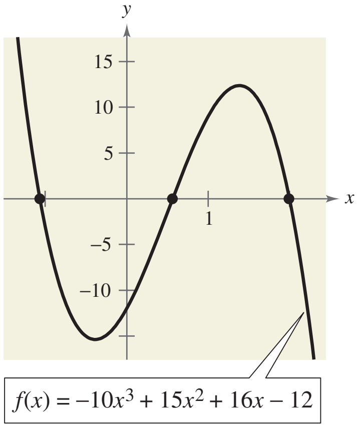


FIGURE 2.32

Conjugate Pairs

In Example 1(c) and (d), note that the pairs of complex zeros are **conjugates.** That is, they are of the form $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only if the polynomial function has *real coefficients*. For instance, the result applies to the function given by $f(x) = x^2 + 1$ but not to the function given by $g(x) = x - i$.

Example 6 Finding a Polynomial with Given Zeros

Find a fourth-degree polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9. \end{aligned}$$

CHECKPOINT Now try Exercise 37.

Factoring a Polynomial

The Linear Factorization Theorem shows that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

However, this result includes the possibility that some of the values of c_i are complex. The following theorem says that even if you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors. For a proof of this theorem, see Proofs in Mathematics on page 214.

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

You may want to remind students that a graphing calculator is helpful in determining real zeros, which in turn are useful in finding the complex zeros.

A quadratic factor with no real zeros is said to be *prime* or **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals*. For example, the quadratic $x^2 + 1 = (x - i)(x + i)$ is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ is irreducible over the rationals but *reducible* over the reals.

Example 7 Finding the Zeros of a Polynomial Function

Find all the zeros of $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$ given that $1 + 3i$ is a zero of f.

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$[x - (1 + 3i)] \quad \text{and} \quad [x - (1 - 3i)]$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r}
x^2 - x - 6 \\
x^2 - 2x + 10 \overline{\smash{\big)}\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\
\underline{x^4 - 2x^3 + 10x^2} \\
-x^3 - 4x^2 + 2x \\
\underline{-x^3 + 2x^2 - 10x} \\
-6x^2 + 12x - 60 \\
\underline{-6x^2 + 12x - 60} \\
0
\end{array}$$

So, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

CHECKPOINT Now try Exercise 47.

Graphical Solution

Because complex zeros always occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. Because the polynomial is a fourth-degree polynomial, you know that there are at most two other zeros of the function. Use a graphing utility to graph

$$y = x^4 - 3x^3 + 6x^2 + 2x - 60$$

as shown in Figure 2.33.

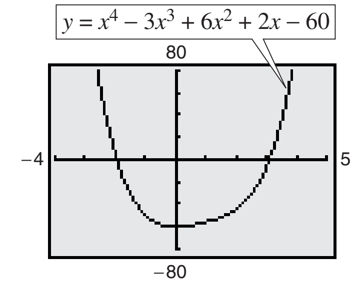


FIGURE 2.33

You can see that -2 and 3 appear to be zeros of the graph of the function. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to confirm that $x = -2$ and $x = 3$ are zeros of the graph. So, you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

In Example 7, if you were not told that $1 + 3i$ is a zero of f, you could still find all zeros of the function by using synthetic division to find the real zeros -2 and 3. Then you could factor the polynomial as $(x + 2)(x - 3)(x^2 - 2x + 10)$. Finally, by using the Quadratic Formula, you could determine that the zeros are $x = -2$, $x = 3$, $x = 1 + 3i$, and $x = 1 - 3i$.

> **STUDY TIP**
>
> In Example 8, the fifth-degree polynomial function has three real zeros. In such cases, you can use the *zoom* and *trace* features or the *zero* or *root* feature of a graphing utility to approximate the real zeros. You can then use these real zeros to determine the complex zeros algebraically.

Example 8 shows how to find all the zeros of a polynomial function, including complex zeros.

Example 8 Finding the Zeros of a Polynomial Function

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all of its zeros.

Solution

The possible rational zeros are $\pm 1, \pm 2, \pm 4$, and ± 8. Synthetic division produces the following.

1	1	0	1	2	−12	8
		1	1	2	4	−8
	1	1	2	4	−8	0

1 is a zero.

−2	1	1	2	4	−8
		−2	2	−8	8
	1	−1	4	−4	0

−2 is a zero.

So, you have

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$$
$$= (x - 1)(x + 2)(x^3 - x^2 + 4x - 4).$$

You can factor $x^3 - x^2 + 4x - 4$ as $(x - 1)(x^2 + 4)$, and by factoring $x^2 + 4$ as

$$x^2 - (-4) = \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right)$$
$$= (x - 2i)(x + 2i)$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1,\ x = 1,\ x = -2,\ x = 2i, \quad \text{and} \quad x = -2i$$

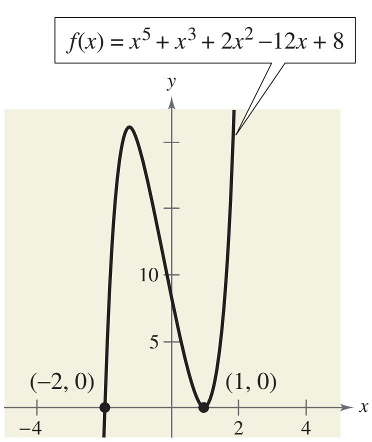


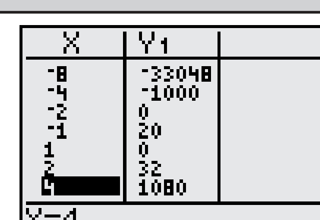
FIGURE 2.34

From the graph of f shown in Figure 2.34, you can see that the *real* zeros are the only ones that appear as x-intercepts. Note that $x = 1$ is a repeated zero.

CHECKPOINT Now try Exercise 63.

Technology

You can use the *table* feature of a graphing utility to help you determine which of the possible rational zeros are zeros of the polynomial in Example 8. The table should be set to *ask* mode. Then enter each of the possible rational zeros in the table. When you do this, you will see that there are two rational zeros, −2 and 1, as shown at the right.



Other Tests for Zeros of Polynomials

You know that an nth-degree polynomial function can have *at most n* real zeros. Of course, many nth-degree polynomials do not have that many real zeros. For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs,** sheds more light on the number of real zeros of a polynomial.

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

A **variation in sign** means that two consecutive coefficients have opposite signs.

When using Descartes's Rule of Signs, a zero of multiplicity k should be counted as k zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

Example 9 Using Descartes's Rule of Signs

Describe the possible real zeros of

$$f(x) = 3x^3 - 5x^2 + 6x - 4.$$

Solution

The original polynomial has *three* variations in sign.

+ to −   + to −

$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

− to +

The polynomial

$$f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4$$
$$= -3x^3 - 5x^2 - 6x - 4$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial $f(x) = 3x^3 - 5x^2 + 6x - 4$ has either three positive real zeros or one positive real zero, and has no negative real zeros. From the graph in Figure 2.35, you can see that the function has only one real zero (it is a positive number, near $x = 1$).

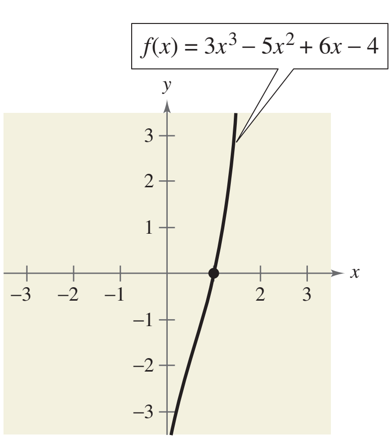


FIGURE 2.35

CHECKPOINT Now try Exercise 79.

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound of the real zeros of f. A real number b is an **upper bound** for the real zeros of f if no zeros are greater than b. Similarly, b is a **lower bound** if no real zeros of f are less than b.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, c is an **upper bound** for the real zeros of f.

2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), c is a **lower bound** for the real zeros of f.

Example 10 Finding the Zeros of a Polynomial Function

Find the real zeros of $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution

The possible real zeros are as follows.

$$\frac{\text{Factors of } 2}{\text{Factors of } 6} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}, \pm 2$$

The original polynomial $f(x)$ has three variations in sign. The polynomial

$$\begin{aligned} f(-x) &= 6(-x)^3 - 4(-x)^2 + 3(-x) - 2 \\ &= -6x^3 - 4x^2 - 3x - 2 \end{aligned}$$

has no variations in sign. As a result of these two findings, you can apply Descartes's Rule of Signs to conclude that there are three positive real zeros or one positive real zero, and no negative zeros. Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. So, you can restrict the search to zeros between 0 and 1. By trial and error, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero.

CHECKPOINT Now try Exercise 87.

In Example 10, notice how the Rational Zero Test, Descartes's Rule of Signs, and the Upper and Lower Bound Rules may be used together in a search for all real zeros of a polynomial function.

Before concluding this section, here are two additional hints that can help you find the real zeros of a polynomial.

1. If the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x$$
$$= x(x^3 - 5x^2 + 3x + 1)$$

you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

2. If you are able to find all but two zeros of $f(x)$, you can always use the Quadratic Formula on the remaining quadratic factor. For instance, if you succeeded in writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x$$
$$= x(x - 1)(x^2 - 4x - 1)$$

you can apply the Quadratic Formula to $x^2 - 4x - 1$ to conclude that the two remaining zeros are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

Example 11 Using a Polynomial Model

You are designing candle-making kits. Each kit contains 25 cubic inches of candle wax and a mold for making a pyramid-shaped candle. You want the height of the candle to be 2 inches less than the length of each side of the candle's square base. What should the dimensions of your candle mold be?

Solution

The volume of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height. The area of the base is x^2 and the height is $(x - 2)$. So, the volume of the pyramid is $V = \frac{1}{3}x^2(x - 2)$. Substituting 25 for the volume yields the following.

$$25 = \frac{1}{3}x^2(x - 2) \qquad \text{Substitute 25 for } V.$$

$$75 = x^3 - 2x^2 \qquad \text{Multiply each side by 3.}$$

$$0 = x^3 - 2x^2 - 75 \qquad \text{Write in general form.}$$

The possible rational solutions are $x = \pm1, \pm3, \pm5, \pm15, \pm25, \pm75$. Use synthetic division to test some of the possible solutions. Note that in this case, it makes sense to test only positive x-values. Using synthetic division, you can determine that $x = 5$ is a solution.

$$\begin{array}{r|rrrr} 5 & 1 & -2 & 0 & -75 \\ & & 5 & 15 & 75 \\ \hline & 1 & 3 & 15 & 0 \end{array}$$

The other two solutions, which satisfy $x^2 + 3x + 15 = 0$, are imaginary and can be discarded. You can conclude that the base of the candle mold should be 5 inches by 5 inches and the height of the mold should be $5 - 2 = 3$ inches.

CHECKPOINT Now try Exercise 107.

Activities

1. Write as a product of linear factors: $f(x) = x^4 - 16$.
   Answer: $(x - 2)(x + 2)(x - 2i)(x + 2i)$
2. Find a third-degree polynomial with integer coefficients that has $2, 3 + i$, and $3 - i$ as zeros.
   Answer: $x^3 - 8x^2 + 22x - 20$
3. Use the zero $x = 2i$ to find all the zeros of $f(x) = x^4 - x^3 - 2x^2 - 4x - 24$.
   Answer: $-2, 3, 2i, -2i$

2.5 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The ________ ________ of ________ states that if $f(x)$ is a polynomial of degree n ($n > 0$), then f has at least one zero in the complex number system.
2. The ________ ________ ________ states that if $f(x)$ is a polynomial of degree n ($n > 0$), then f has precisely n linear factors $f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$ where $c_1, c_2, \ldots, c_n$ are complex numbers.
3. The test that gives a list of the possible rational zeros of a polynomial function is called the ________ ________ Test.
4. If $a + bi$ is a complex zero of a polynomial with real coefficients, then so is its ________, $a - bi$.
5. A quadratic factor that cannot be factored further as a product of linear factors containing real numbers is said to be ________ over the ________.
6. The theorem that can be used to determine the possible numbers of positive real zeros and negative real zeros of a function is called ________ ________ of ________.
7. A real number b is a(n) ________ bound for the real zeros of f if no real zeros are less than b, and is a(n) ________ bound if no real zeros are greater than b.

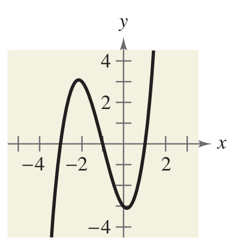
PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

In Exercises 1–6, find all the zeros of the function.

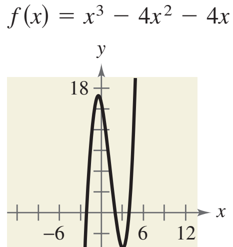
1. $f(x) = x(x - 6)^2$
2. $f(x) = x^2(x + 3)(x^2 - 1)$
3. $g(x) = (x - 2)(x + 4)^3$
4. $f(x) = (x + 5)(x - 8)^2$
5. $f(x) = (x + 6)(x + i)(x - i)$
6. $h(t) = (t - 3)(t - 2)(t - 3i)(t + 3i)$

In Exercises 7–10, use the Rational Zero Test to list all possible rational zeros of *f*. Verify that the zeros of *f* shown on the graph are contained in the list.

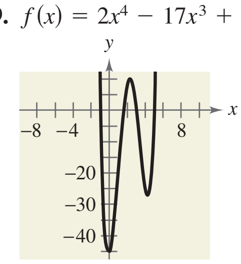
7. $f(x) = x^3 + 3x^2 - x - 3$



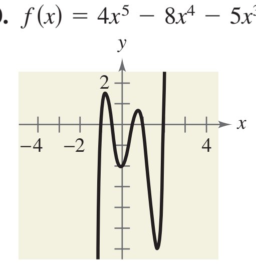
8. $f(x) = x^3 - 4x^2 - 4x + 16$



9. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$



10. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$



In Exercises 11–20, find all the rational zeros of the function.

11. $f(x) = x^3 - 6x^2 + 11x - 6$
12. $f(x) = x^3 - 7x - 6$
13. $g(x) = x^3 - 4x^2 - x + 4$
14. $h(x) = x^3 - 9x^2 + 20x - 12$
15. $h(t) = t^3 + 12t^2 + 21t + 10$
16. $p(x) = x^3 - 9x^2 + 27x - 27$
17. $C(x) = 2x^3 + 3x^2 - 1$
18. $f(x) = 3x^3 - 19x^2 + 33x - 9$
19. $f(x) = 9x^4 - 9x^3 - 58x^2 + 4x + 24$
20. $f(x) = 2x^4 - 15x^3 + 23x^2 + 15x - 25$

In Exercises 21–24, find all real solutions of the polynomial equation.

21. $z^4 - z^3 - 2z - 4 = 0$

22. $x^4 - 13x^2 - 12x = 0$

23. $2y^4 + 7y^3 - 26y^2 + 23y - 6 = 0$

24. $x^5 - x^4 - 3x^3 + 5x^2 - 2x = 0$

In Exercises 25–28, (a) list the possible rational zeros of *f*, (b) sketch the graph of *f* so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of *f*.

25. $f(x) = x^3 + x^2 - 4x - 4$

26. $f(x) = -3x^3 + 20x^2 - 36x + 16$

27. $f(x) = -4x^3 + 15x^2 - 8x - 3$

28. $f(x) = 4x^3 - 12x^2 - x + 15$

In Exercises 29–32, (a) list the possible rational zeros of *f*, (b) use a graphing utility to graph *f* so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of *f*.

29. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

30. $f(x) = 4x^4 - 17x^2 + 4$

31. $f(x) = 32x^3 - 52x^2 + 17x + 3$

32. $f(x) = 4x^3 + 7x^2 - 11x - 18$

Graphical Analysis **In Exercises 33–36, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places, (b) determine one of the exact zeros (use synthetic division to verify your result), and (c) factor the polynomial completely.**

33. $f(x) = x^4 - 3x^2 + 2$

34. $P(t) = t^4 - 7t^2 + 12$

35. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

36. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

In Exercises 37–42, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

37. $1, 5i, -5i$

38. $4, 3i, -3i$

39. $6, -5 + 2i, -5 - 2i$

40. $2, 4 + i, 4 - i$

41. $\frac{2}{3}, -1, 3 + \sqrt{2}i$

42. $-5, -5, 1 + \sqrt{3}i$

In Exercises 43–46, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

43. $f(x) = x^4 + 6x^2 - 27$

44. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$
(*Hint:* One factor is $x^2 - 6$.)

45. $f(x) = x^4 - 4x^3 + 5x^2 - 2x - 6$
(*Hint:* One factor is $x^2 - 2x - 2$.)

46. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$
(*Hint:* One factor is $x^2 + 4$.)

In Exercises 47–54, use the given zero to find all the zeros of the function.

	Function	*Zero*
47.	$f(x) = 2x^3 + 3x^2 + 50x + 75$	$5i$
48.	$f(x) = x^3 + x^2 + 9x + 9$	$3i$
49.	$f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$	$2i$
50.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
51.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
52.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
53.	$f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$	$-3 + \sqrt{2}i$
54.	$f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$

In Exercises 55–72, find all the zeros of the function and write the polynomial as a product of linear factors.

55. $f(x) = x^2 + 25$

56. $f(x) = x^2 - x + 56$

57. $h(x) = x^2 - 4x + 1$

58. $g(x) = x^2 + 10x + 23$

59. $f(x) = x^4 - 81$

60. $f(y) = y^4 - 625$

61. $f(z) = z^2 - 2z + 2$

62. $h(x) = x^3 - 3x^2 + 4x - 2$

63. $g(x) = x^3 - 6x^2 + 13x - 10$

64. $f(x) = x^3 - 2x^2 - 11x + 52$

65. $h(x) = x^3 - x + 6$

66. $h(x) = x^3 + 9x^2 + 27x + 35$

67. $f(x) = 5x^3 - 9x^2 + 28x + 6$

68. $g(x) = 3x^3 - 4x^2 + 8x + 8$

69. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

70. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

71. $f(x) = x^4 + 10x^2 + 9$

72. $f(x) = x^4 + 29x^2 + 100$

In Exercises 73–78, find all the zeros of the function. When there is an extended list of possible rational zeros, use a graphing utility to graph the function in order to discard any rational zeros that are obviously not zeros of the function.

73. $f(x) = x^3 + 24x^2 + 214x + 740$

74. $f(s) = 2s^3 - 5s^2 + 12s - 5$

75. $f(x) = 16x^3 - 20x^2 - 4x + 15$

76. $f(x) = 9x^3 - 15x^2 + 11x - 5$

77. $f(x) = 2x^4 + 5x^3 + 4x^2 + 5x + 2$

78. $g(x) = x^5 - 8x^4 + 28x^3 - 56x^2 + 64x - 32$

In Exercises 79–86, use Descartes's Rule of Signs to determine the possible numbers of positive and negative zeros of the function.

79. $g(x) = 5x^5 + 10x$

80. $h(x) = 4x^2 - 8x + 3$

81. $h(x) = 3x^4 + 2x^2 + 1$

82. $h(x) = 2x^4 - 3x + 2$

83. $g(x) = 2x^3 - 3x^2 - 3$

84. $f(x) = 4x^3 - 3x^2 + 2x - 1$

85. $f(x) = -5x^3 + x^2 - x + 5$

86. $f(x) = 3x^3 + 2x^2 + x + 3$

In Exercises 87–90, use synthetic division to verify the upper and lower bounds of the real zeros of *f*.

87. $f(x) = x^4 - 4x^3 + 15$
(a) Upper: $x = 4$ (b) Lower: $x = -1$

88. $f(x) = 2x^3 - 3x^2 - 12x + 8$
(a) Upper: $x = 4$ (b) Lower: $x = -3$

89. $f(x) = x^4 - 4x^3 + 16x - 16$
(a) Upper: $x = 5$ (b) Lower: $x = -3$

90. $f(x) = 2x^4 - 8x + 3$
(a) Upper: $x = 3$ (b) Lower: $x = -4$

In Exercises 91–94, find all the real zeros of the function.

91. $f(x) = 4x^3 - 3x - 1$

92. $f(z) = 12z^3 - 4z^2 - 27z + 9$

93. $f(y) = 4y^3 + 3y^2 + 8y + 6$

94. $g(x) = 3x^3 - 2x^2 + 15x - 10$

In Exercises 95–98, find all the rational zeros of the polynomial function.

95. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

96. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

97. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

98. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

In Exercises 99–102, match the cubic function with the numbers of rational and irrational zeros.

(a) Rational zeros: 0; irrational zeros: 1
(b) Rational zeros: 3; irrational zeros: 0
(c) Rational zeros: 1; irrational zeros: 2
(d) Rational zeros: 1; irrational zeros: 0

99. $f(x) = x^3 - 1$

100. $f(x) = x^3 - 2$

101. $f(x) = x^3 - x$

102. $f(x) = x^3 - 2x$

103. ***Geometry*** An open box is to be made from a rectangular piece of material, 15 centimeters by 9 centimeters, by cutting equal squares from the corners and turning up the sides.

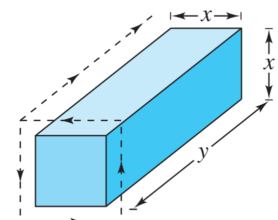
(a) Let x represent the length of the sides of the squares removed. Draw a diagram showing the squares removed from the original piece of material and the resulting dimensions of the open box.

(b) Use the diagram to write the volume V of the box as a function of x. Determine the domain of the function.

(c) Sketch the graph of the function and approximate the dimensions of the box that will yield a maximum volume.

(d) Find values of x such that $V = 56$. Which of these values is a physical impossibility in the construction of the box? Explain.

104. ***Geometry*** A rectangular package to be sent by a delivery service (see figure) can have a maximum combined length and girth (perimeter of a cross section) of 120 inches.



(a) Show that the volume of the package is

$$V(x) = 4x^2(30 - x).$$

(b) Use a graphing utility to graph the function and approximate the dimensions of the package that will yield a maximum volume.

(c) Find values of x such that $V = 13{,}500$. Which of these values is a physical impossibility in the construction of the package? Explain.

105. ***Advertising Cost*** A company that produces MP3 players estimates that the profit P (in dollars) for selling a particular model is given by

$$P = -76x^3 + 4830x^2 - 320{,}000, \quad 0 \le x \le 60$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, find the smaller of two advertising amounts that will yield a profit of \$2,500,000.

106. ***Advertising Cost*** A company that manufactures bicycles estimates that the profit P (in dollars) for selling a particular model is given by

$$P = -45x^3 + 2500x^2 - 275{,}000, \quad 0 \le x \le 50$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, find the smaller of two advertising amounts that will yield a profit of \$800,000.

107. ***Geometry*** A bulk food storage bin with dimensions 2 feet by 3 feet by 4 feet needs to be increased in size to hold five times as much food as the current bin. (Assume each dimension is increased by the same amount.)

(a) Write a function that represents the volume V of the new bin.

(b) Find the dimensions of the new bin.

108. ***Geometry*** A rancher wants to enlarge an existing rectangular corral such that the total area of the new corral is 1.5 times that of the original corral. The current corral's dimensions are 250 feet by 160 feet. The rancher wants to increase each dimension by the same amount.

(a) Write a function that represents the area A of the new corral.

(b) Find the dimensions of the new corral.

(c) A rancher wants to add a length to the sides of the corral that are 160 feet, and twice the length to the sides that are 250 feet, such that the total area of the new corral is 1.5 times that of the original corral. Repeat parts (a) and (b). Explain your results.

109. ***Cost*** The ordering and transportation cost C (in thousands of dollars) for the components used in manufacturing a product is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where x is the order size (in hundreds). In calculus, it can be shown that the cost is a minimum when

$$3x^3 - 40x^2 - 2400x - 36{,}000 = 0.$$

Use a calculator to approximate the optimal order size to the nearest hundred units.

110. ***Height of a Baseball*** A baseball is thrown upward from a height of 6 feet with an initial velocity of 48 feet per second, and its height h (in feet) is

$$h(t) = -16t^2 + 48t + 6, \quad 0 \leq t \leq 3$$

where t is the time (in seconds). You are told the ball reaches a height of 64 feet. Is this possible?

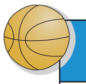
111. ***Profit*** The demand equation for a certain product is $p = 140 - 0.0001x$, where p is the unit price (in dollars) of the product and x is the number of units produced and sold. The cost equation for the product is $C = 80x + 150{,}000$, where C is the total cost (in dollars) and x is the number of units produced. The total profit obtained by producing and selling x units is

$$P = R - C = xp - C.$$

You are working in the marketing department of the company that produces this product, and you are asked to determine a price p that will yield a profit of 9 million dollars. Is this possible? Explain.

Model It

112. ***Athletics*** The attendance A (in millions) at NCAA women's college basketball games for the years 1997 through 2003 is shown in the table, where t represents the year, with $t = 7$ corresponding to 1997. (Source: National Collegiate Athletic Association)



Year, t	Attendance, A
7	6.7
8	7.4
9	8.0
10	8.7
11	8.8
12	9.5
13	10.2

(a) Use the *regression* feature of a graphing utility to find a cubic model for the data.

(b) Use the graphing utility to create a scatter plot of the data. Then graph the model and the scatter plot in the same viewing window. How do they compare?

(c) According to the model found in part (a), in what year did attendance reach 8.5 million?

(d) According to the model found in part (a), in what year did attendance reach 9 million?

(e) According to the right-hand behavior of the model, will the attendance continue to increase? Explain.

Synthesis

True or False? **In Exercises 113 and 114, decide whether the statement is true or false. Justify your answer.**

113. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

114. If $x = -i$ is a zero of the function given by

$$f(x) = x^3 + ix^2 + ix - 1$$

then $x = i$ must also be a zero of f.

Think About It **In Exercises 115–120, determine (if possible) the zeros of the function g if the function f has zeros at $x = r_1, x = r_2$, and $x = r_3$.**

115. $g(x) = -f(x)$

116. $g(x) = 3f(x)$

117. $g(x) = f(x - 5)$

118. $g(x) = f(2x)$

119. $g(x) = 3 + f(x)$

120. $g(x) = f(-x)$

121. ***Exploration*** Use a graphing utility to graph the function given by $f(x) = x^4 - 4x^2 + k$ for different values of k. Find values of k such that the zeros of f satisfy the specified characteristics. (Some parts do not have unique answers.)

(a) Four real zeros

(b) Two real zeros, each of multiplicity 2

(c) Two real zeros and two complex zeros

(d) Four complex zeros

122. ***Think About It*** Will the answers to Exercise 121 change for the function g?

(a) $g(x) = f(x - 2)$ (b) $g(x) = f(2x)$

123. ***Think About It*** A third-degree polynomial function f has real zeros $-2, \frac{1}{2}$, and 3, and its leading coefficient is negative. Write an equation for f. Sketch the graph of f. How many different polynomial functions are possible for f?

124. ***Think About It*** Sketch the graph of a fifth-degree polynomial function whose leading coefficient is positive and that has one zero at $x = 3$ of multiplicity 2.

125. ***Writing*** Compile a list of all the various techniques for factoring a polynomial that have been covered so far in the text. Give an example illustrating each technique, and write a paragraph discussing when the use of each technique is appropriate.

126. Use the information in the table to answer each question.

Interval	Value of $f(x)$
$(-\infty, -2)$	Positive
$(-2, 1)$	Negative
$(1, 4)$	Negative
$(4, \infty)$	Positive

(a) What are the three real zeros of the polynomial function f?

(b) What can be said about the behavior of the graph of f at $x = 1$?

(c) What is the least possible degree of f? Explain. Can the degree of f ever be odd? Explain.

(d) Is the leading coefficient of f positive or negative? Explain.

(e) Write an equation for f. (There are many correct answers.)

(f) Sketch a graph of the equation you wrote in part (e).

127. (a) Find a quadratic function f (with integer coefficients) that has $\pm\sqrt{b}i$ as zeros. Assume that b is a positive integer.

(b) Find a quadratic function f (with integer coefficients) that has $a \pm bi$ as zeros. Assume that b is a positive integer.

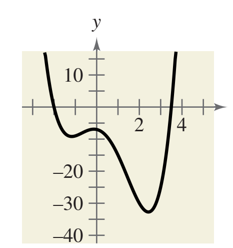
128. ***Graphical Reasoning*** The graph of one of the following functions is shown below. Identify the function shown in the graph. Explain why each of the others is not the correct function. Use a graphing utility to verify your result.

(a) $f(x) = x^2(x + 2)(x - 3.5)$

(b) $g(x) = (x + 2)(x - 3.5)$

(c) $h(x) = (x + 2)(x - 3.5)(x^2 + 1)$

(d) $k(x) = (x + 1)(x + 2)(x - 3.5)$



Skills Review

In Exercises 129–132, perform the operation and simplify.

129. $(-3 + 6i) - (8 - 3i)$

130. $(12 - 5i) + 16i$

131. $(6 - 2i)(1 + 7i)$

132. $(9 - 5i)(9 + 5i)$

In Exercises 133–138, use the graph of *f* to sketch the graph of *g*. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

133. $g(x) = f(x - 2)$

134. $g(x) = f(x) - 2$

135. $g(x) = 2f(x)$

136. $g(x) = f(-x)$

137. $g(x) = f(2x)$

138. $g(x) = f\left(\frac{1}{2}x\right)$