1.10 Mathematical Modeling and Variation

What you should learn

- Use mathematical models to approximate sets of data points.
- Use the *regression* feature of a graphing utility to find the equation of a least squares regression line.
- Write mathematical models for direct variation.
- Write mathematical models for direct variation as an *n*th power.
- Write mathematical models for inverse variation.
- Write mathematical models for joint variation.

Why you should learn it

You can use functions as models to represent a wide variety of real-life data sets. For instance, in Exercise 71 on page 113, a variation model can be used to model the water temperature of the ocean at various depths.

Introduction

You have already studied some techniques for fitting models to data. For instance, in Section 1.3, you learned how to find the equation of a line that passes through two points. In this section, you will study other techniques for fitting models to data: *least squares regression* and *direct and inverse variation*. The resulting models are either polynomial functions or rational functions. (Rational functions will be studied in Chapter 2.)

Example 1 **A Mathematical Model**

The numbers of insured commercial banks y (in thousands) in the United States for the years 1996 to 2001 are shown in the table. (Source: Federal Deposit Insurance Corporation)

Year	Insured commercial banks, y
1996	9.53
1997	9.14
1998	8.77
1999	8.58
2000	8.32
2001	8.08

A linear model that approximates the data is $y = -0.283t + 11.14$ for $6 \le t \le 11$, where t is the year, with $t = 6$ corresponding to 1996. Plot the actual data *and* the model on the same graph. How closely does the model represent the data?

Solution

The actual data are plotted in Figure 1.100, along with the graph of the linear model. From the graph, it appears that the model is a "good fit" for the actual data. You can see how well the model fits by comparing the actual values of y with the values of y given by the model. The values given by the model are labeled y^* in the table below.

t	6	7	8	9	10	11
y	9.53	9.14	8.77	8.58	8.32	8.08
y^*	9.44	9.16	8.88	8.59	8.31	8.03

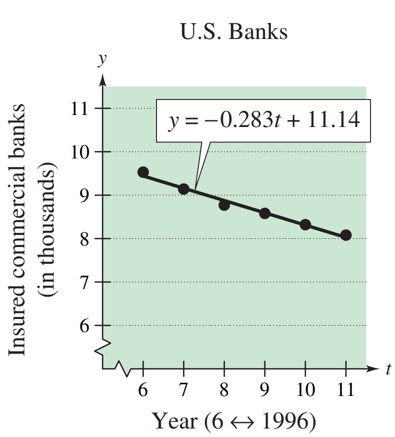


FIGURE 1.100

CHECKPOINT Now try Exercise 1.

Note in Example 1 that you could have chosen any two points to find a line that fits the data. However, the given linear model was found using the *regression* feature of a graphing utility and is the line that *best* fits the data. This concept of a "best-fitting" line is discussed on the next page.

Least Squares Regression and Graphing Utilities

This subsection along with Exercises 7–10 can be omitted if technology is not used in your course.

So far in this text, you have worked with many different types of mathematical models that approximate real-life data. In some instances the model was given (as in Example 1), whereas in other instances you were asked to find the model using simple algebraic techniques or a graphing utility.

To find a model that approximates the data most accurately, statisticians use a measure called the **sum of square differences,** which is the sum of the squares of the differences between actual data values and model values. The "best-fitting" linear model, called the **least squares regression line,** is the one with the least sum of square differences. Recall that you can approximate this line visually by plotting the data points and drawing the line that appears to fit best—or you can enter the data points into a calculator or computer and use the *linear regression* feature of the calculator or computer. When you use the *regression* feauture of a graphing calculator or computer program, you will notice that the program may also output an "r-value." This r-value is the **correlation coefficient** of the data and gives a measure of how well the model fits the data. The closer the value of $|r|$ is to 1, the better the fit.

Example 2 Finding a Least Squares Regression Line

The amounts p (in millions of dollars) of total annual prize money awarded at the Indianapolis 500 race from 1995 to 2004 are shown in the table. Construct a scatter plot that represents the data and find the least squares regression line for the data. (Source: indy500.com)

Year	Prize money, p
1995	8.06
1996	8.11
1997	8.61
1998	8.72
1999	9.05
2000	9.48
2001	9.61
2002	10.03
2003	10.15
2004	10.25

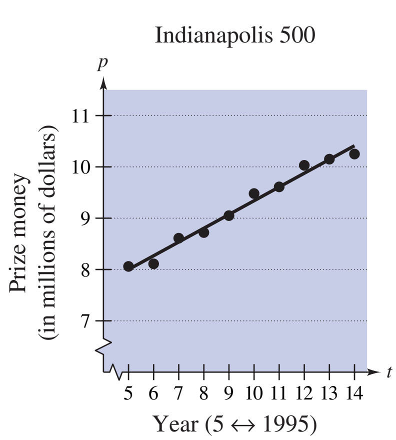


FIGURE 1.101

t	p	$p*$
5	8.06	8.00
6	8.11	8.27
7	8.61	8.54
8	8.72	8.80
9	9.05	9.07
10	9.48	9.34
11	9.61	9.61
12	10.03	9.88
13	10.15	10.14
14	10.25	10.41

Solution

Let $t = 5$ represent 1995. The scatter plot for the points is shown in Figure 1.101. Using the *regression* feature of a graphing utility, you can determine that the equation of the least squares regression line is

$$p = 0.268t + 6.66.$$

To check this model, compare the actual p-values with the p-values given by the model, which are labeled $p*$ in the table at the left. The correlation coefficient for this model is $r \approx 0.991$, which implies that the model is a good fit.

CHECKPOINT Now try Exercise 7.

Activity

In physics, Hooke's Law for springs states that the distance a spring is stretched or compressed from its natural, or equilibrium, length varies directly as the force on the spring. Distance is measured in inches (or meters) and force is measured in pounds (or newtons). One newton is equivalent to about 0.225 pound.

a. Apply the concept of direct variation to find an equation relating the distance stretched (or compressed) to the force applied.

Answer: $d = kF$, where d is the distance stretched or compressed, F is the force, and k is the "spring constant."

b. A force of 100 newtons stretches a spring 0.25 meter. How far will a force of 80 newtons stretch the spring?

Answer: 0.20 meter

Direct Variation

There are two basic types of linear models. The more general model has a y-intercept that is nonzero.

$$y = mx + b, \quad b \neq 0$$

The simpler model

$$y = kx$$

has a y-intercept that is zero. In the simpler model, y is said to **vary directly** as x, or to be **directly proportional** to x.

Direct Variation

The following statements are equivalent.

1. y **varies directly** as x.

2. y is **directly proportional** to x.

3. $y = kx$ for some nonzero constant k.

k is the **constant of variation** or the **constant of proportionality.**

Example 3 Direct Variation

In Pennsylvania, the state income tax is directly proportional to *gross income*. You are working in Pennsylvania and your state income tax deduction is \$46.05 for a gross monthly income of \$1500. Find a mathematical model that gives the Pennsylvania state income tax in terms of gross income.

Solution

Verbal Model: State income tax $=$ k $\cdot$ Gross income

Labels:
State income tax $= y$ (dollars)
Gross income $= x$ (dollars)
Income tax rate $= k$ (percent in decimal form)

Equation: $y = kx$

To solve for k, substitute the given information into the equation $y = kx$, and then solve for k.

$y = kx$	Write direct variation model.
$46.05 = k(1500)$	Substitute $y = 46.05$ and $x = 1500$.
$0.0307 = k$	Simplify.

So, the equation (or model) for state income tax in Pennsylvania is

$$y = 0.0307x.$$

In other words, Pennsylvania has a state income tax rate of 3.07% of gross income. The graph of this equation is shown in Figure 1.102.

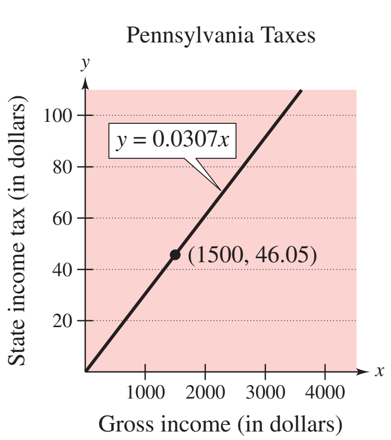


FIGURE 1.102

CHECKPOINT Now try Exercise 33.

Direct Variation as an *n*th Power

Another type of direct variation relates one variable to a *power* of another variable. For example, in the formula for the area of a circle

$$A = \pi r^2$$

the area A is directly proportional to the square of the radius r. Note that for this formula, π is the constant of proportionality.

STUDY TIP

Note that the direct variation model $y = kx$ is a special case of $y = kx^n$ with $n = 1$.

Direct Variation as an *n*th Power

The following statements are equivalent.

1. y **varies directly as the *n*th power** of x.
2. y is **directly proportional to the *n*th power** of x.
3. $y = kx^n$ for some constant k.

Example 4 Direct Variation as *n*th Power

The distance a ball rolls down an inclined plane is directly proportional to the square of the time it rolls. During the first second, the ball rolls 8 feet. (See Figure 1.103.)

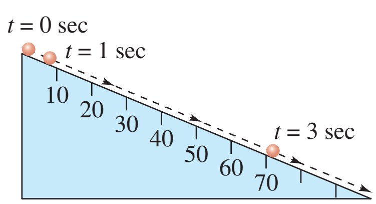


FIGURE 1.103

a. Write an equation relating the distance traveled to the time.

b. How far will the ball roll during the first 3 seconds?

Solution

a. Letting d be the distance (in feet) the ball rolls and letting t be the time (in seconds), you have

$$d = kt^2.$$

Now, because $d = 8$ when $t = 1$, you can see that $k = 8$, as follows.

$$d = kt^2$$
$$8 = k(1)^2$$
$$8 = k$$

So, the equation relating distance to time is

$$d = 8t^2.$$

b. When $t = 3$, the distance traveled is $d = 8(3)^2 = 8(9) = 72$ feet.

CHECKPOINT Now try Exercise 63.

In Examples 3 and 4, the direct variations are such that an *increase* in one variable corresponds to an *increase* in the other variable. This is also true in the model $d = \frac{1}{5}F$, $F > 0$, where an increase in F results in an increase in d. You should not, however, assume that this always occurs with direct variation. For example, in the model $y = -3x$, an increase in x results in a *decrease* in y, and yet y is said to vary directly as x.

Inverse Variation

> ### Inverse Variation
>
> The following statements are equivalent.
>
> **1.** y **varies inversely** as x. **2.** y is **inversely proportional** to x.
>
> **3.** $y = \dfrac{k}{x}$ for some constant k.

If x and y are related by an equation of the form $y = k/x^n$, then y varies inversely as the nth power of x (or y is inversely proportional to the nth power of x).

Some applications of variation involve problems with *both* direct and inverse variation in the same model. These types of models are said to have **combined variation.**

Example 5 Direct and Inverse Variation

A gas law states that the volume of an enclosed gas varies directly as the temperature *and* inversely as the pressure, as shown in Figure 1.104. The pressure of a gas is 0.75 kilogram per square centimeter when the temperature is 294 K and the volume is 8000 cubic centimeters. (a) Write an equation relating pressure, temperature, and volume. (b) Find the pressure when the temperature is 300 K and the volume is 7000 cubic centimeters.

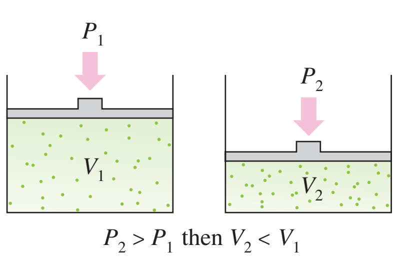


FIGURE 1.104 *If the temperature is held constant and pressure increases, volume decreases.*

Solution

a. Let V be volume (in cubic centimeters), let P be pressure (in kilograms per square centimeter), and let T be temperature (in Kelvin). Because V varies directly as T and inversely as P, you have

$$V = \frac{kT}{P}.$$

Now, because $P = 0.75$ when $T = 294$ and $V = 8000$, you have

$$8000 = \frac{k(294)}{0.75}$$

$$k = \frac{6000}{294} = \frac{1000}{49}.$$

So, the equation relating pressure, temperature, and volume is

$$V = \frac{1000}{49}\left(\frac{T}{P}\right).$$

b. When $T = 300$ and $V = 7000$, the pressure is

$$P = \frac{1000}{49}\left(\frac{300}{7000}\right) = \frac{300}{343} \approx 0.87 \text{ kilogram per square centimeter.}$$

CHECKPOINT Now try Exercise 65.

Joint Variation

In Example 5, note that when a direct variation and an inverse variation occur in the same statement, they are coupled with the word "and." To describe two different *direct* variations in the same statement, the word **jointly** is used.

Joint Variation

The following statements are equivalent.

1. z **varies jointly** as x and y.
2. z is **jointly proportional** to x and y.
3. $z = kxy$ for some constant k.

If x, y, and z are related by an equation of the form

$$z = kx^n y^m$$

then z varies jointly as the nth power of x and the mth power of y.

Example 6 Joint Variation

The *simple* interest for a certain savings account is jointly proportional to the time and the principal. After one quarter (3 months), the interest on a principal of \$5000 is \$43.75.

a. Write an equation relating the interest, principal, and time.

b. Find the interest after three quarters.

Solution

a. Let $I =$ interest (in dollars), $P =$ principal (in dollars), and $t =$ time (in years). Because I is jointly proportional to P and t, you have

$$I = kPt.$$

For $I = 43.75$, $P = 5000$, and $t = \frac{1}{4}$, you have

$$43.75 = k(5000)\left(\frac{1}{4}\right)$$

which implies that $k = 4(43.75)/5000 = 0.035$. So, the equation relating interest, principal, and time is

$$I = 0.035Pt$$

which is the familiar equation for simple interest where the constant of proportionality, 0.035, represents an annual interest rate of 3.5%.

b. When $P = \$5000$ and $t = \frac{3}{4}$, the interest is

$$I = (0.035)(5000)\left(\frac{3}{4}\right)$$

$$= \$131.25.$$

CHECKPOINT Now try Exercise 67.

Activities

1. Find a mathematical model for "a is jointly proportional to y and z."

   Answer: $a = kyz$

2. Determine the constant of proportionality if x is inversely proportional to the square of y, when $y = 7$ and $x = 3$.

   Answer: 147

3. Write a sentence using variation terminology to describe the formula $F = k\dfrac{m_1 m_2}{t^2}$.

   Answer: F is jointly proportional to m_1 and m_2 and inversely proportional to the square of t.

1.10 Exercises

VOCABULARY CHECK: Fill in the blanks.

1. Two techniques for fitting models to data are called direct ________ and least squares ________.
2. Statisticians use a measure called ________ of________ ________ to find a model that approximates a set of data most accurately.
3. An r-value of a set of data, also called a ________ ________, gives a measure of how well a model fits a set of data.
4. Direct variation models can be described as y varies directly as x, or y is ________ ________ to x.
5. In direct variation models of the form $y = kx$, k is called the ________ of ________.
6. The direct variation model $y = kx^n$ can be described as y varies directly as the nth power of x, or y is ________ ________ to the nth power of x.
7. The mathematical model $y = \dfrac{k}{x}$ is an example of ________ variation.
8. Mathematical models that involve both direct and inverse variation are said to have ________ variation.
9. The joint variation model $z = kxy$ can be described as z varies jointly as x and y, or z is ________ ________ to x and y.

PREREQUISITE SKILLS REVIEW: Practice and review algebra skills needed for this section at **www.Eduspace.com.**

1. ***Employment*** The total numbers of employees (in thousands) in the United States from 1992 to 2002 are given by the following ordered pairs.

(1992, 128,105)	(1998, 137,673)
(1993, 129,200)	(1999, 139,368)
(1994, 131,056)	(2000, 142,583)
(1995, 132,304)	(2001, 143,734)
(1996, 133,943)	(2002, 144,683)
(1997, 136,297)	

A linear model that approximates the data is $y = 1767.0t + 123{,}916$, where y represents the number of employees (in thousands) and $t = 2$ represents 1992. Plot the actual data and the model on the same set of coordinate axes. How closely does the model represent the data? (Source: U.S. Bureau of Labor Statistics)

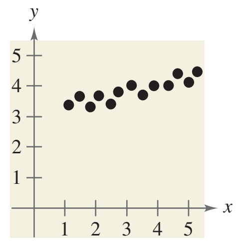
2. ***Sports*** The winning times (in minutes) in the women's 400-meter freestyle swimming event in the Olympics from 1948 to 2004 are given by the following ordered pairs.

(1948, 5.30)	(1980, 4.15)
(1952, 5.20)	(1984, 4.12)
(1956, 4.91)	(1988, 4.06)
(1960, 4.84)	(1992, 4.12)
(1964, 4.72)	(1996, 4.12)
(1968, 4.53)	(2000, 4.10)
(1972, 4.32)	(2004, 4.09)
(1976, 4.16)	

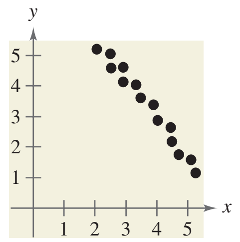
A linear model that approximates the data is $y = -0.022t + 5.03$, where y represents the winning time (in minutes) and $t = 0$ represents 1950. Plot the actual data and the model on the same set of coordinate axes. How closely does the model represent the data? Does it appear that another type of model may be a better fit? Explain. (Source: *The World Almanac and Book of Facts*)

In Exercises 3–6, sketch the line that you think best approximates the data in the scatter plot. Then find an equation of the line. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

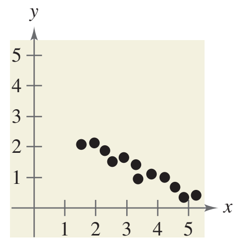
3.



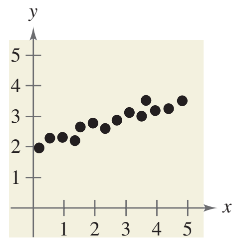
4.



5.



6.



7. ***Sports*** The lengths (in feet) of the winning men's discus throws in the Olympics from 1912 to 2004 are listed below. (Source: *The World Almanac and Book of Facts*)

1912	148.3	1952	180.5	1980	218.7
1920	146.6	1956	184.9	1984	218.5
1924	151.3	1960	194.2	1988	225.8
1928	155.3	1964	200.1	1992	213.7
1932	162.3	1968	212.5	1996	227.7
1936	165.6	1972	211.3	2000	227.3
1948	173.2	1976	221.5	2004	229.3

(a) Sketch a scatter plot of the data. Let y represent the length of the winning discus throw (in feet) and let $t = 12$ represent 1912.

(b) Use a straightedge to sketch the best-fitting line through the points and find an equation of the line.

(c) Use the *regression* feature of a graphing utility to find the least squares regression line that fits the data.

(d) Compare the linear model you found in part (b) with the linear model given by the graphing utility in part (c).

(e) Use the models from parts (b) and (c) to estimate the winning men's discus throw in the year 2008.

(f) Use your school's library, the Internet, or some other reference source to analyze the accuracy of the estimate in part (e).

8. ***Revenue*** The total revenues (in millions of dollars) for Outback Steakhouse from 1995 to 2003 are listed below. (Source: Outback Steakhouse, Inc.)

1995	664.0	2000	1906.0
1996	937.4	2001	2127.0
1997	1151.6	2002	2362.1
1998	1358.9	2003	2744.4
1999	1646.0		

(a) Sketch a scatter plot of the data. Let y represent the total revenue (in millions of dollars) and let $t = 5$ represent 1995.

(b) Use a straightedge to sketch the best-fitting line through the points and find an equation of the line.

(c) Use the *regression* feature of a graphing utility to find the least squares regression line that fits the data.

(d) Compare the linear model you found in part (b) with the linear model given by the graphing utility in part (c).

(e) Use the models from parts (b) and (c) to estimate the revenues of Outback Steakhouse in 2005.

(f) Use your school's library, the Internet, or some other reference source to analyze the accuracy of the estimate in part (e).

9. ***Data Analysis: Broadway Shows*** The table shows the annual gross ticket sales S (in millions of dollars) for Broadway shows in New York City from 1995 through 2004. (Source: The League of American Theatres and Producers, Inc.)

Year	Sales, S
1995	406
1996	436
1997	499
1998	558
1999	588
2000	603
2001	666
2002	643
2003	721
2004	771

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 5$ represent 1995.

(b) Use the *regression* feature of a graphing utility to find the equation of the least squares regression line that fits the data.

(c) Use the graphing utility to graph the scatter plot you found in part (a) and the model you found in part (b) in the same viewing window. How closely does the model represent the data?

(d) Use the model to estimate the annual gross ticket sales in 2005 and 2007.

(e) Interpret the meaning of the slope of the linear model in the context of the problem.

10. ***Data Analysis: Television Households*** The table shows the numbers x (in millions) of households with cable television and the numbers y (in millions) of households with color television sets in the United States from 1995 through 2002. (Source: Nielson Media Research; Television Bureau of Advertising, Inc.)

Households with cable, x	Households with color TV, y
63	94
65	95
66	97
67	98
75	99
77	101
80	102
86	105

(a) Use the *regression* feature of a graphing utility to find the equation of the least squares regression line that fits the data.

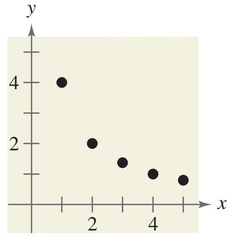
(b) Use the graphing utility to create a scatter plot of the data. Then graph the model you found in part (a) and the scatter plot in the same viewing window. How closely does the model represent the data?

(c) Use the model to estimate the number of households with color television sets if the number of households with cable television is 90 million.

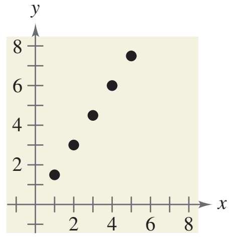
(d) Interpret the meaning of the slope of the linear model in the context of the problem.

Think About It **In Exercises 11 and 12, use the graph to determine whether y varies directly as some power of x or inversely as some power of x. Explain.**

11.



12.



In Exercises 13–16, use the given value of k to complete the table for the direct variation model $y = kx^2$. Plot the points on a rectangular coordinate system.

x	2	4	6	8	10
$y = kx^2$					

13. $k = 1$ **14.** $k = 2$

15. $k = \frac{1}{2}$ **16.** $k = \frac{1}{4}$

In Exercises 17–20, use the given value of k to complete the table for the inverse variation model

$$y = \frac{k}{x^2}.$$

Plot the points on a rectangular coordinate system.

x	2	4	6	8	10
$y = \dfrac{k}{x^2}$					

17. $k = 2$ **18.** $k = 5$

19. $k = 10$ **20.** $k = 20$

In Exercises 21–24, determine whether the variation model is of the form $y = kx$ or $y = k/x$, and find k.

21.

x	5	10	15	20	25
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

22.

x	5	10	15	20	25
y	2	4	6	8	10

23.

x	5	10	15	20	25
y	-3.5	-7	-10.5	-14	-17.5

24.

x	5	10	15	20	25
y	24	12	8	6	$\frac{24}{5}$

Direct Variation **In Exercises 25–28, assume that y is directly proportional to x. Use the given x-value and y-value to find a linear model that relates y and x.**

25. $x = 5,\ y = 12$

26. $x = 2,\ y = 14$

27. $x = 10,\ y = 2050$

28. $x = 6,\ y = 580$

29. ***Simple Interest*** The simple interest on an investment is directly proportional to the amount of the investment. By investing \$2500 in a certain bond issue, you obtained an interest payment of \$87.50 after 1 year. Find a mathematical model that gives the interest I for this bond issue after 1 year in terms of the amount invested P.

30. ***Simple Interest*** The simple interest on an investment is directly proportional to the amount of the investment. By investing \$5000 in a municipal bond, you obtained an interest payment of \$187.50 after 1 year. Find a mathematical model that gives the interest I for this municipal bond after 1 year in terms of the amount invested P.

31. ***Measurement*** On a yardstick with scales in inches and centimeters, you notice that 13 inches is approximately the same length as 33 centimeters. Use this information to find a mathematical model that relates centimeters to inches. Then use the model to find the numbers of centimeters in 10 inches and 20 inches.

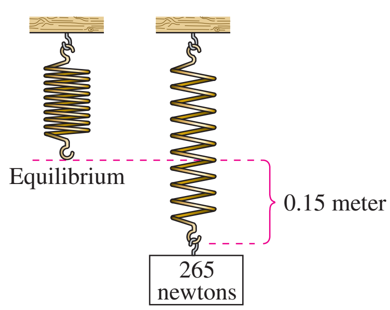
32. ***Measurement*** When buying gasoline, you notice that 14 gallons of gasoline is approximately the same amount of gasoline as 53 liters. Then use this information to find a linear model that relates gallons to liters. Then use the model to find the numbers of liters in 5 gallons and 25 gallons.

33. ***Taxes*** Property tax is based on the assessed value of a property. A house that has an assessed value of \$150,000 has a property tax of \$5520. Find a mathematical model that gives the amount of property tax y in terms of the assessed value x of the property. Use the model to find the property tax on a house that has an assessed value of \$200,000.

34. ***Taxes*** State sales tax is based on retail price. An item that sells for \$145.99 has a sales tax of \$10.22. Find a mathematical model that gives the amount of sales tax y in terms of the retail price x. Use the model to find the sales tax on a \$540.50 purchase.

Hooke's Law **In Exercises 35–38, use Hooke's Law for springs, which states that the distance a spring is stretched (or compressed) varies directly as the force on the spring.**

35. A force of 265 newtons stretches a spring 0.15 meter (see figure).



(a) How far will a force of 90 newtons stretch the spring?

(b) What force is required to stretch the spring 0.1 meter?

36. A force of 220 newtons stretches a spring 0.12 meter. What force is required to stretch the spring 0.16 meter?

37. The coiled spring of a toy supports the weight of a child. The spring is compressed a distance of 1.9 inches by the weight of a 25-pound child. The toy will not work properly if its spring is compressed more than 3 inches. What is the weight of the heaviest child who should be allowed to use the toy?

38. An overhead garage door has two springs, one on each side of the door (see figure). A force of 15 pounds is required to stretch each spring 1 foot. Because of a pulley system, the springs stretch only one-half the distance the door travels. The door moves a total of 8 feet, and the springs are at their natural length when the door is open. Find the combined lifting force applied to the door by the springs when the door is closed.

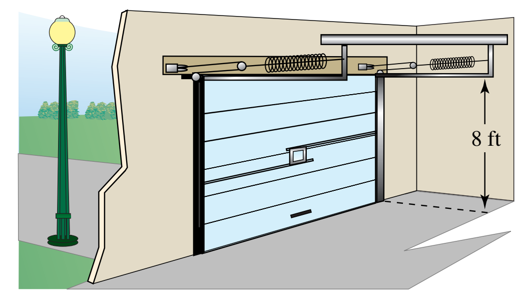


FIGURE FOR 38

In Exercises 39–48, find a mathematical model for the verbal statement.

39. A varies directly as the square of r.

40. V varies directly as the cube of e.

41. y varies inversely as the square of x.

42. h varies inversely as the square root of s.

43. F varies directly as g and inversely as r^2.

44. z is jointly proportional to the square of x and the cube of y.

45. ***Boyle's Law:*** For a constant temperature, the pressure P of a gas is inversely proportional to the volume V of the gas.

46. ***Newton's Law of Cooling:*** The rate of change R of the temperature of an object is proportional to the difference between the temperature T of the object and the temperature T_e of the environment in which the object is placed.

47. ***Newton's Law of Universal Gravitation:*** The gravitational attraction F between two objects of masses m_1 and m_2 is proportional to the product of the masses and inversely proportional to the square of the distance r between the objects.

48. ***Logistic Growth:*** The rate of growth R of a population is jointly proportional to the size S of the population and the difference between S and the maximum population size L that the environment can support.

In Exercises 49–54, write a sentence using the variation terminology of this section to describe the formula.

49. ***Area of a triangle:*** $A = \frac{1}{2}bh$

50. ***Surface area of a sphere:*** $S = 4\pi r^2$

51. ***Volume of a sphere:*** $V = \frac{4}{3}\pi r^3$

52. ***Volume of a right circular cylinder:*** $V = \pi r^2 h$

53. ***Average speed:*** $r = \dfrac{d}{t}$

54. ***Free vibrations:*** $\omega = \sqrt{\dfrac{kg}{W}}$

In Exercises 55–62, find a mathematical model representing the statement. (In each case, determine the constant of proportionality.)

55. A varies directly as r^2. ($A = 9\pi$ when $r = 3$.)

56. y varies inversely as x. ($y = 3$ when $x = 25$.)

57. y is inversely proportional to x. ($y = 7$ when $x = 4$.)

58. z varies jointly as x and y. ($z = 64$ when $x = 4$ and $y = 8$.)

59. F is jointly proportional to r and the third power of s. ($F = 4158$ when $r = 11$ and $s = 3$.)

60. P varies directly as x and inversely as the square of y. $\left(P = \frac{28}{3}\right.$ when $x = 42$ and $\left.y = 9.\right)$

61. z varies directly as the square of x and inversely as y. ($z = 6$ when $x = 6$ and $y = 4$.)

62. v varies jointly as p and q and inversely as the square of s. ($v = 1.5$ when $p = 4.1$, $q = 6.3$, and $s = 1.2$.)

Ecology **In Exercises 63 and 64, use the fact that the diameter of the largest particle that can be moved by a stream varies approximately directly as the square of the velocity of the stream.**

63. A stream with a velocity of $\frac{1}{4}$ mile per hour can move coarse sand particles about 0.02 inch in diameter. Approximate the velocity required to carry particles 0.12 inch in diameter.

64. A stream of velocity v can move particles of diameter d or less. By what factor does d increase when the velocity is doubled?

Resistance **In Exercises 65 and 66, use the fact that the resistance of a wire carrying an electrical current is directly proportional to its length and inversely proportional to its cross-sectional area.**

65. If #28 copper wire (which has a diameter of 0.0126 inch) has a resistance of 66.17 ohms per thousand feet, what length of #28 copper wire will produce a resistance of 33.5 ohms?

66. A 14-foot piece of copper wire produces a resistance of 0.05 ohm. Use the constant of proportionality from Exercise 65 to find the diameter of the wire.

67. ***Work*** The work W (in joules) done when lifting an object varies jointly with the mass m (in kilograms) of the object and the height h (in meters) that the object is lifted. The work done when a 120-kilogram object is lifted 1.8 meters is 2116.8 joules. How much work is done when lifting a 100-kilogram object 1.5 meters?

68. ***Spending*** The prices of three sizes of pizza at a pizza shop are as follows.

9-inch: \$8.78, 12-inch: \$11.78, 15-inch: \$14.18

You would expect that the price of a certain size of pizza would be directly proportional to its surface area. Is that the case for this pizza shop? If not, which size of pizza is the best buy?

69. ***Fluid Flow*** The velocity v of a fluid flowing in a conduit is inversely proportional to the cross-sectional area of the conduit. (Assume that the volume of the flow per unit of time is held constant.) Determine the change in the velocity of water flowing from a hose when a person places a finger over the end of the hose to decrease its cross-sectional area by 25%.

70. ***Beam Load*** The maximum load that can be safely supported by a horizontal beam varies jointly as the width of the beam and the square of its depth, and inversely as the length of the beam. Determine the changes in the maximum safe load under the following conditions.

(a) The width and length of the beam are doubled.

(b) The width and depth of the beam are doubled.

(c) All three of the dimensions are doubled.

(d) The depth of the beam is halved.

Model It

71. ***Data Analysis: Ocean Temperatures*** An oceanographer took readings of the water temperatures C (in degrees Celsius) at several depths d (in meters). The data collected are shown in the table.

Depth, d	Temperature, C
1000	4.2°
2000	1.9°
3000	1.4°
4000	1.2°
5000	0.9°

(a) Sketch a scatter plot of the data.

(b) Does it appear that the data can be modeled by the inverse variation model $C = k/d$? If so, find k for each pair of coordinates.

(c) Determine the mean value of k from part (b) to find the inverse variation model $C = k/d$.

(d) Use a graphing utility to plot the data points and the inverse model in part (c).

(e) Use the model to approximate the depth at which the water temperature is 3°C.

72. ***Data Analysis: Physics Experiment*** An experiment in a physics lab requires a student to measure the compressed lengths y (in centimeters) of a spring when various forces of F pounds are applied. The data are shown in the table.

Force, F	Length, y
0	0
2	1.15
4	2.3
6	3.45
8	4.6
10	5.75
12	6.9

(a) Sketch a scatter plot of the data.

(b) Does it appear that the data can be modeled by Hooke's Law? If so, estimate k. (See Exercises 35–38.)

(c) Use the model in part (b) to approximate the force required to compress the spring 9 centimeters.

73. ***Data Analysis: Light Intensity*** A light probe is located x centimeters from a light source, and the intensity y (in microwatts per square centimeter) of the light is measured. The results are shown as ordered pairs (x, y).

(30, 0.1881)	(34, 0.1543)	(38, 0.1172)
(42, 0.0998)	(46, 0.0775)	(50, 0.0645)

A model for the data is $y = 262.76/x^{2.12}$.

(a) Use a graphing utility to plot the data points and the model in the same viewing window.

(b) Use the model to approximate the light intensity 25 centimeters from the light source.

74. ***Illumination*** The illumination from a light source varies inversely as the square of the distance from the light source. When the distance from a light source is doubled, how does the illumination change? Discuss this model in terms of the data given in Exercise 73. Give a possible explanation of the difference.

Synthesis

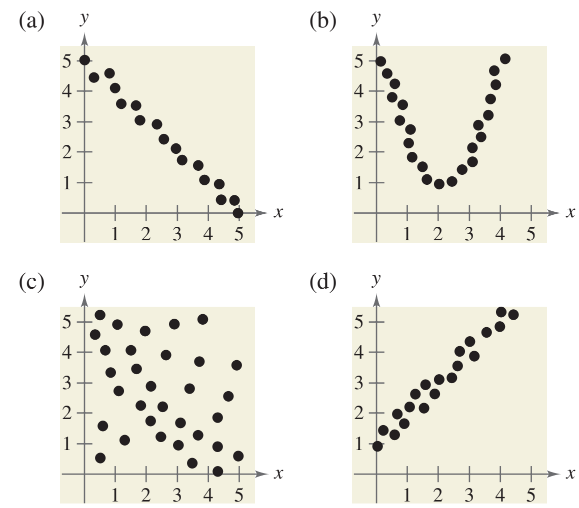
True or False? **In Exercises 75–77, decide whether the statement is true or false. Justify your answer.**

75. If y varies directly as x, then if x increases, y will increase as well.

76. In the equation for kinetic energy, $E = \frac{1}{2}mv^2$, the amount of kinetic energy E is directly proportional to the mass m of an object and the square of its velocity v.

77. If the correlation coefficient for a least squares regression line is close to -1, the regression line cannot be used to describe the data.

78. Discuss how well the data shown in each scatter plot can be approximated by a linear model.



79. ***Writing*** A linear mathematical model for predicting prize winnings at a race is based on data for 3 years. Write a paragraph discussing the potential accuracy or inaccuracy of such a model.

80. ***Research Project*** Use your school's library, the Internet, or some other reference source to find data that you think describe a linear relationship. Create a scatter plot of the data and find the least squares regression line that represents the data points. Interpret the slope and y-intercept in the context of the data. Write a summary of your findings.

Skills Review

In Exercises 81–84, solve the inequality and graph the solution on the real number line.

81. $3x + 2 > 17$

82. $-7x + 10 \leq -1 + x$

83. $|2x - 1| < 9$

84. $|4 - 3x| + 7 \geq 12$

In Exercises 85 and 86, evaluate the function at each value of the independent variable and simplify.

85. $f(x) = \dfrac{x^2 + 5}{x - 3}$

(a) $f(0)$ (b) $f(-3)$ (c) $f(4)$

86. $f(x) = \begin{cases} -x^2 + 10, & x \geq -2 \\ 6x^2 - 1, & x < -2 \end{cases}$

(a) $f(-2)$ (b) $f(1)$ (c) $f(-8)$

87. Make a Decision To work an extended application analyzing registered voters in United States, visit this text's website at *college.hmco.com*. *(Data Source: U.S. Census Bureau)*